ON THE FORMS

OF

PLANE QUARTIC CURVES.

A DISSERTATION

PRESENTED TO THE FACULTY OF BRYN MAWR COLLEGE
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

RUTH GENTRY.

NEW YORK:
PRESS OF ROBERT DRUMMOND.
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LIFE.

I was born in Stilesville, Indiana, February 22, 1862. My early education was received in the public schools of Stilesville. I was graduated from the Indiana State Normal School in 1880 and spent the next ten years in preparatory and college teaching, and in study in the University of Michigan, from which I was graduated with the degree Ph.B. in 1890. In 1890–91, and again in 1892–93, I held the Fellowship in Mathematics in Bryn Mawr College. During the year 1891–92 I held the European Fellowship of the Association of Collegiate Alumnæ and studied in the University of Berlin. For the first semester of the year 1892–93 I attended lectures at the Sorbonne, and then returned to Bryn Mawr College, where I remained until June, 1894; during the year 1893–94 I was Fellow by Courtesy. In June, 1894, I passed the examinations for the degree of Doctor of Philosophy in Bryn Mawr College, my Major subject being Pure Mathematics; and my Minors, Physics and Applied Mathematics, which I studied under Professors Mackenzie and Harkness and Dr. Buckingham.

My graduate study in Mathematics was pursued under Professors Scott and Harkness, of Bryn Mawr College, and Professor Fuchs, of Berlin University. I also attended lectures by Professor Schwarz and Dr. Schlesinger, of Berlin University, and Professors Picard, Darboux, and Raffy, of the Sorbonne. To all of these my acknowledgments are due; and to Professor Scott, under whose direction this paper was written, I find it hard adequately to express my indebtedness and my gratitude for the inspiration derived from her unfailing interest in my work.

CONTENTS.

ERRATA.

Page 14, line 14, for uz read nz.

" 22, " 20, for α^2x^2 read α^2y^2.

" 42, " 24, omit $\because$

" 45, " 27, for $\underline{\equiv}$ read $\equiv$

ON THE FORMS

OF

PLANE QUARTIC CURVES.

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1. MANY papers dealing with curves of the fourth order, or Quartic Curves, are to be found in the various mathematical periodicals; but these leave the actual appearance of the curve as a whole so largely to the reader's imagination that it is here proposed to give a complete enumeration of the fundamental forms of Plane Quartic Curves as they appear when projected so as to cut the line infinity the least possible number of times,* together with evidence that the forms presented can exist.

2. Since the appearance of a curve depends mainly upon the number and nature of its multiple points, multiple tangents, inflexions, and circuits, a satisfactory point of departure is afforded by the following table, computed by applying Plücker's equations, and then Klein's equation,

$$m + I + 2T = n + K + 2A \dagger$$

(which gives columns T and I), to the quartic.

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* Figs. 2, 6, Plate I, are not so drawn, but Figs. 2′, 6′ show the types when thus projected.

† *Math. Ann.*, x, p. 199; 1876.

EXPLANATION OF THE TABLE.

$\delta =$ total number of nodes;
$\kappa =$ total number of cusps ;
$m =$ order;
$n =$ class;
$\tau =$ total number of bitangents ;
$i =$ total number of inflexions ;
$p =$ deficiency ;
$K =$ number of real cusps;
$A =$ number of distinct, isolated dps, i.e., acnodes ;
$T =$ number of real dts with imaginary contacts, i.e., isolated dts; *
$I =$ number of real inflexions.

The maximum number of circuits being always one greater than p (see Harnack, *Math. Ann.*, x, p. 189; 1876), it has not been deemed necessary to indicate this number in the table.

Table of Possible Combinations of Point and Line Singularities in Plane Quartic Curves, as deduced from Plücker's Equations and Klein's Equation.

I. NON-SINGULAR QUARTICS. [$p = 3$]

$n = 12, \quad \tau = 28, \quad i = 24:$

	T.	I.
(1)	0	8
(2)	1	6
(3)	2	4
(4)	3	2
(5)	4	0

* Throughout this paper *isolated dt* means a *real* dt with *imaginary* contacts; *non-isolated dt*, a *real* dt with *real* contacts.

II. UNICURSAL QUARTICS.† [$p = 0$]

$$\delta = 3, \quad n = 6, \quad \tau = 4, \quad i = 6:$$

	A.	T.	I.	
(1)	0	0	2	1. 3 real, distinct crunodes. 2. 1 crunode, 1 tacnode. 3. 1 oscnode. 4. 1 3-pt of 1st kind. 5. 1 real crunode, 2 imaginary nodes.
(2)		1	0	
(3)	1	0	4	1. 2 real, distinct crunodes, 1 acnode. 2. 1 tacnode, 1 acnode. 3. 1 3-pt of 4th kind. 4. 2 imaginary nodes, 1 acnode.
(4)		1	2	
*(5)		2	0	
(6)	2	0	6	1 crunode, 2 acnodes.
(7)		1	4	
(8)		2	2	
*(9)		3	0	
(10)	3	1	6	3 acnodes.
(11)		2	4	
(12)		3	2	
(13)		4	0	

$$\delta = 2, \quad \kappa = 1, \quad n = 5, \quad \tau = 2, \quad i = 4, \quad K = 1:$$

	A.	T.	I.	
(14)	0	0	2	1. 2 real, distinct crunodes, 1 cusp.‡ 2. 1 tacnode, 1 cusp. 3. 1 node-cusp, 1 crunode. 4. 1 tacnode-cusp. 5. 1 3-pt of 2d kind. 6. 2 imaginary nodes, 1 cusp.
(15)		1	0	
(16)	1	0	4	1. 1 crunode, 1 cusp, 1 acnode. 2. 1 node-cusp, 1 acnode.
(17)		1	2	
*(18)		2	0	

† The four grand divisions of Quartics are given in the table in the order in which they are discussed in this paper. (See Arts. 9, 73.)

* For all forms so marked in the table see note A.

‡ *Cusp* is used throughout this paper to denote an ordinary simple cusp.

	K.	A.	T.	I.	
(19)		2	1	4	1 cusp, 2 acnodes.
(20)			2	2	

$\delta = 1, \quad \kappa = 2, \quad n = 4, \quad \tau = 1, \quad i = 2:$

	K.	A.	T.	I.	
(21)	2	0	0	2	1. 1 crunode, 2 real cusps. 2. 1 node-cusp, 1 cusp. 3. 1 3-pt of 3d kind.
(22)			1	0	
(23)	2	1	1	2	2 real cusps, 1 acnode.
(24)	0	0	0	0	1 crunode, 2 imaginary cusps.
(25)	0	1	0	2	2 imaginary cusps, 1 acnode.
(26)			1	0	

$\kappa = 3, \quad n = 3, \quad \tau = 1, \quad i = 0:$

	K.	A.	T.	I.	
(27)	3	0	1	0	3 real cusps.
(28)	1	0	0	0	1 real cusp, 2 imaginary cusps.

III. QUARTICS WITH ONE DOUBLE POINT. [$p = 2$]

$\delta = 1, \quad n = 10, \quad \tau = 16, \quad i = 18:$

	A.	T.	I.	
(1)	0	0	6	1 crunode
(2)		1	4	
(3)		2	2	
(4)		3	0	
(5)	1	0	8	1 acnode.
(6)		1	6	
(7)		2	4	
(8)		3	2	
(9)		4	0	

$\kappa = 1, \quad n = 9, \quad \tau = 10, \quad i = 16, \quad K = 1:$

	T.	I.	
(10)	0	6	1 cusp.
(11)	1	4	
(12)	2	2	
(13)	3	0	

IV. QUARTICS WITH TWO DOUBLE POINTS. [$p = 1$]

$\delta = 2, \quad n = 8, \quad \tau = 8, \quad i = 12:$

	A.	T.	I.	
(1)	0	0	4	1. 2 real distinct crunodes.
(2)		1	2	2. 1 tacnode.
(3)		2	0	3. 2 imaginary nodes.
(4)	1	0	6	1 crunode, 1 acnode.
(5)		1	4	
(6)		2	2	
*(7)		3	0	
(8)	2	0	8	2 acnodes.
(9)		1	6	
(10)		2	4	
(11)		3	2	
(12)		4	0	

$\delta = 1, \quad \kappa = 1, \quad n = 7, \quad \tau = 4, \quad i = 10, \quad K = 1:$

	A.	T.	I.	
(13)	0	0	4	1. 1 crunode, 1 cusp.
(14)		1	2	2. 1 node-cusp.
*(15)		2	0	
(16)	1	0	6	1 cusp, 1 acnode.
(17)		1	4	
(18)		2	2	
*(19)		3	0	

* For all forms so marked in the table see note A.

$$\kappa = 2, \quad n = 6, \quad \tau = 1, \quad i = 8:$$

	K.	*A.*	*T.*	*I.*	
(20)	2	0	0	4	2 real cusps.
(21)			1	2	
(22)	0	0	0	2	2 imaginary cusps.
(23)			1	0	

3. For completeness the accepted analysis of the higher singularities possible in a quartic curve is here given.

A *tacnode* is composed of two consecutive nodes; the tangent at a tacnode counts as *two* consecutive, non-isolated dts.

A *node-cusp,* or *ramphoid cusp,* is composed of one crunode and one cusp in consecutive position and involves one real inflexion; the tangent counts as *one* non-isolated dt, and in its consecutive position also as the inflexional tangent at the real inflexion involved in the singularity.

An *oscnode* is composed of three consecutive nodes, either on a right line or on the same parabola of curvature. Since the latter, usually distinguished as *curved* oscnode, is the only kind possible in a curve of lower order than the sixth, it will here be called *oscnode* simply; its tangent counts as *three* consecutive, non-isolated dts.

A *tacnode-cusp* is composed of a node-cusp and a crunode,—that is, of two crunodes and one cusp in consecutive position,—and involves one real inflexion; the tangent counts as *two* consecutive, non-isolated dts and as one inflexional tangent.

A *triple point of the first kind* is composed of *three coincident crunodes* formed by the crossing of three ordinary branches with real, distinct tangents.

A *triple point of the second kind* is composed of one cusp and two crunodes in coincidence, formed by the passage of an ordinary branch through a cusp; the tangents are all real, two of them consecutive, forming the cuspidal tangent.

A *triple point of the third kind* is composed of one crunode and two cusps, coincident; the three tangents are real and consecutive.

A *triple point of the fourth kind* is composed of three coincident nodes formed by the passage of an ordinary branch through an acnode; one tangent is real, two are imaginary.

For curves of higher order than the fourth, subdivisions of the kinds of

triple point are possible according to the number of real inflexions that may be situated at the triple point, but in a quartic there can be no inflexion at a triple point, since the tangent already meets the curve there in four points.

4. The Plückerian characteristics of a curve are irrespective of reality and coincidence, but Klein does not say that his equation is applicable to the case of consecutive or coincident double points,* nor that any *possibility* as given by his equation necessarily involves the existence of a curve of the kind in question. It is therefore to be investigated, among other things, how many and which of the tabular possibilities are realities in the case of any given type of quartic. These questions will be considered in their proper place.

I. NON-SINGULAR QUARTICS.†

5. Zeuthen ‡ has so fully discussed this genus that only a summary of his results and a rough drawing of figures for the different types will here be attempted.

Zeuthen uses the real bitangents of a quartic as a basis of classification. These he divides into bitangents of the first and second species. Bitangents of the first species are (a) real bitangents with real contacts which lie on the same circuit; (b) real bitangents with imaginary contacts; that is, isolated bitangents. Bitangents of the second kind are real bitangents with contacts on different circuits. Every non-singular quartic has four and only four bitangents of the first species; these, intersecting in the plane, form four triangles and three quadrilaterals. According to situation with reference to these triangles and quadrilaterals the curves in question are divided by Zeuthen into three main divisions:

Annular quartics, composed of two circuits, one inside the other.

Quadrilateral quartics, composed of three circuits at most, distributed in the three quadrilaterals formed by the four bitangents of the first species.

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* This equation has been considered from a purely algebraic point of view by Brill (Ueber Singularitäten ebener algebraischen Curven, *Math. Ann.*, XVI, pp. 351, 388; 1879).

† See Plate I.

‡ Sur les différentes formes des courbes du quatrième ordre (*Math. Ann.* VII, pp. 410–432; 1874).

Trilateral quartics, composed of four circuits at most, distributed in the four triangles formed by the bitangents of the first kind.

The annular quartic is the only one that can have more than one circuit in the same triangle or quadrilateral, and it always consists solely of the two circuits, one inside the other.

6. If t, u, v, w be the four bitangents described in the foregoing, and ϕ a conic through their eight points of contact (such a conic is always possible), the quartic may be written

$$tuvw = k\phi^2.$$

This conic ϕ may take thirteen different positions, giving thirty-six species of non-singular quartics. A circuit with no indentation is called an *oval;* one with a single indentation, and therefore two real inflexions, is called a *unifolium;* one with two indentations is called a *bifolium,* etc. In the description of each species the *maximum* number of ovals is given; there may be fewer, but at least some one quartic of the species can be found having the maximum number. How many may disappear at once Zeuthen does not discuss.

The following are his thirty-six species: *

i. *ϕ cuts all four sides of a quadrilateral.*

Species 1. Annular quartic, composed of one *quadrifolium* and one *oval* inside.

Species 2. Quadrilateral quartic, composed of one *quadrifolium* and two ovals.

Species 3. Trilateral quartic, composed of four *unifolia.*

ii. *ϕ cuts three sides of one quadrilateral and one side of another.*

Species 4. Quadrilateral quartic, composed of one *trifolium,* one *unifolium,* and one oval.

Species 5. Trilateral quartic, composed of one *bifolium,* two *unifolia,* and one oval.

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* See Plate I.

iii. *ϕ cuts two sides of two quadrilaterals.*

Species 6. Quadrilateral quartic, composed of two *bifolia* and one oval.

Species 7. Trilateral quartic, composed of two *bifolia* and two ovals.

iv. *ϕ cuts two sides of one quadrilateral and one side of each of the other two.* The two sides of the first quadrilateral are necessarily *opposite* sides.

Species 8. Quadrilateral quartic, composed of one *bifolium* and two *unifolia.*

Species 9. Trilateral quartic, composed of one *trifolium,* one *unifolium,* and two ovals.

v. *ϕ cuts three sides of one quadrilateral.*

Species 10. Annular quartic, composed of one *trifolium* and one *oval* inside.

Species 11. Quadrilateral quartic, composed of one *trifolium* and two ovals.

Species 12. Trilateral quartic, composed of three *unifolia* and one oval.

vi. *ϕ cuts two opposite sides of one quadrilateral and one side of another.*

Species 13. Quadrilateral quartic, composed of one *bifolium,* one *unifolium,* and one oval.

Species 14. Trilateral quartic, composed of one *bifolium,* one *unifolium,* and two ovals.

vii. *ϕ cuts two consecutive sides of one quadrilateral and one side of that quadrilateral which has for a pair of* opposite *sides the pair (consecutive) which are cut in the first quadrilateral.*

Species 15. Quadrilateral quartic of description 13.

Species 16. Trilateral quartic of description 14.

viii. *φ cuts three sides of a triangle* (*or one side of each quadrilateral*).

Species 17. Annular quartic, composed of one *trifolium* and one *oval* inside.

Species 18. Quadrilateral quartic, composed of three *unifolia.*

Species 19. Trilateral quartic, composed of one *trifolium* and three ovals.

ix. *φ cuts two opposite sides of a quadrilateral.*

Species 20. Annular quartic, composed of one *bifolium* and one *oval* inside.

Species 21. Quadrilateral quartic, composed of one *bifolium* and two ovals.

Species 22. Trilateral quartic, composed of two *unifolia* and two ovals.

x. *φ cuts two consecutive sides of a quadrilateral.*

Species 23. Annular quartic of description 20.

Species 24. Quadrilateral quartic of description 21.

Species 25. Trilateral quartic of description 22.

xi. *φ cuts two sides of one triangle.*

Species 26. Annular quartic, composed of one *bifolium* and one *oval* inside.

Species 27. Quadrilateral quartic, composed of two *unifolia* and one oval.

Species 28. Trilateral quartic, composed of one *bifolium* and three ovals.

xii. *φ cuts one side of a triangle* (*or quadrilateral*).

Species 29, 30. Annular quartic, situated either in a quadrilateral or in a triangle and composed of one *unifolium* and one *oval* inside.

Species 31. Quadrilateral quartic, composed of one *unifolium* and two ovals.

Species 32. Trilateral quartic, composed of one *unifolium* and three ovals.

xiii. *ϕ does not cut any of the four lines in real points.*

Species 33, 34. Annular quartic, situated in a quadrilateral or in a triangle, and composed of one *oval* inside *another.*

Species 35. Quadrilateral quartic, composed of three *ovals.*

Species 36. Trilateral quartic, composed of four *ovals.*

Comparing these with the table (Art. 2, I), the correspondence is seen to be as follows:

T.	*I.*	Zeuthen's Species.
0	8	1.... 9
1	6	10....19
2	4	20....28
3	2	29....32
4	0	33....36

7. In the equation

$$tuvw = k\phi^2$$

Zeuthen takes no account of the nature of ϕ, which may be a proper or degenerate conic, real or imaginary. If it be imaginary, its intersections with the real bitangents being then all imaginary,* we have simply case xiii. A little examination shows that the case of ϕ a pair of right lines gives no forms of the non-singular quartic not given by ϕ a proper conic. Again, since any proper conic may be projected into any other, there is no loss of generality in taking for ϕ the conic best suited to the purpose of bringing the quartic into the finite part of the plane.

8. Two particular cases not considered by Zeuthen perhaps deserve special mention; viz., when ϕ *touches* one or more of the four bitangents t, u, v, w; and when three or all four of these bitangents are concurrent.†

A point of contact of ϕ with one of the bitangents gives rise to a point of undulation on the quartic: that is, a point in which two

* An imaginary conic of the form $u + iv$ has of course four real points, and might therefore have real points on a dt of the quartic; but if $\phi \equiv u + iv$ the quartic has imaginary coefficients and is excluded from consideration.

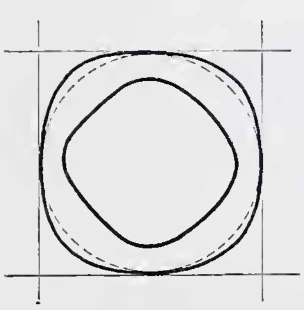
† Zeuthen considers coincidence of bitangents as a means of obtaining singular quartics, and the point of intersection of coincident bitangents with another bitangent is of course a point of concurrence; but he does not consider the concurrence of distinct bitangents.

real inflexions are consecutive and the two points of contact of the bitangents are consecutive. A bitangent in such position is of the first species with real contacts, but may be regarded as just about to pass off into an *isolated* bitangent. Since real contacts must become coincident (algebraically; often *consecutive* geometrically) in order to pass to imaginary, *whenever a special bitangent of the first kind may be either isolated or non-isolated in a given type of curve the form with an undulation may be expected as an intermediate form.* To the eye a point of undulation differs from an ordinary point only in the fact that the curve sits closer on its tangent; hence this form will not ordinarily be shown in a special figure, but it is to be understood as existing in every case, as a transition form between two curves whose only essential difference is that a bitangent that is non-isolated in the one is isolated in the other. Since a quartic can have but four bitangents of the first species, the curve cannot have more than four undulations: that four are simultaneously possible is evident from the fact that the conic ϕ may touch all four of the bitangents t, u, v, w in the position i, (Art. **6**). A simple example is

$$(x^2 - 1)(y^2 - 1) = k(x^2 + y^2 - 1)^2;$$

that is,

$$t \equiv x + 1, \quad u \equiv x - 1, \quad v \equiv y - 1, \quad w \equiv y + 1, \quad \phi \equiv x^2 + y^2 - 1;$$

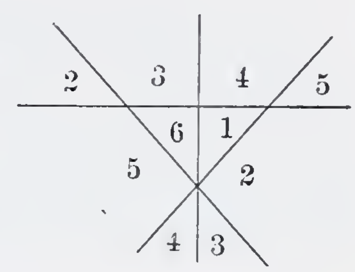


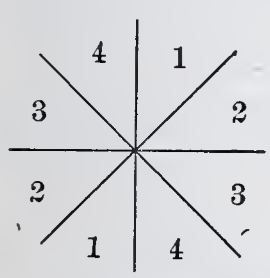
which for $k > 1$ is annular, for $k = 1$ is acnodal, and for $k < 1$ is a quadrilateral quartic.

The concurrence of three or all of the bitangents t, u, v, w has only an excluding effect on the general forms if the point of concurrence be not on ϕ,—a condition which is necessarily satisfied since, if the point of concurrence be on ϕ, the curve is no longer non-singular. The distinction of trilateral and quadrilateral is obliterated, and the quartic cannot have more than three real circuits if three of these bitangents are concurrent, nor more than two if all four are concurrent. This readily appears from the following considerations:—the curve

$$tuvw = k\phi^2$$

cannot have two circuits external one to the other in the same division of the plane, for the same reason that makes this impossible in the case of non-concurrent bitangents (Zeuthen, *loc. cit.*, p. 416, § 10); it cannot lie in two compartments separated by an odd number of these bitangents, since crossing one of them changes the sign of the left member of the equation while leaving the right member unchanged in sign. Now, when three of these bitangents concur the plane is divided into six triangles, from any one of which we may pass to two and only two others by crossing an even number of these bitangents; hence the curve may lie in three compartments, viz., 1, 3, 5 or 2, 4, 6. When all four of the bitangents in question concur the plane is divided into four compartments, from any one of which we may pass to but one other by crossing an even number of these bitangents, thus giving two compartments, viz., 1, 3 or 2, 4, for the curve.





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9. On pages 424–428 of the paper already referred to Zeuthen gives methods for obtaining the possible forms of singular quartics from his thirty-six nonsingular species. The methods are the shrinking of ovals into isolated points ; the joining of the inner oval of an annular quartic to the outer circuit ; the joining of a single circuit to itself ; the passing of ϕ through one or more vertices of the 4-line figure formed by the bitangents of the first species; the coincidence of bitangents of the first species; and combinations of these processes. Instead of proceeding to find by these methods the forms of quartics with one dp, then two, then three, we pass at once to the unicursal curves, which are perhaps the most important of the three grand divisions of singular quartics, and treat them analytically before considering the curves having only one or two dps.

10. In the study of forms of singular quartics the arrangement of the divisions and subdivisions of classes II–IV, as shown in the table (Art. 2), does not always commend itself for purposes of analytical investigation, and in the following pages the forms will be discussed in such order as seems most convenient for the method

of investigation employed; the table will serve the double purpose of obviating the necessity of inquiring into the existence of forms which, as shown by the table, could not exist, and of providing a check on results obtained from examination of equations.

II. UNICURSAL QUARTICS.*

11. The three dps may be all real, or two may be conjugate imaginaries. The case when all three are real will be first considered: in this case the dps may be all distinct; two consecutive; all three consecutive; or all three coincident. (See analysis of singularities, Art. **3**.)

§ I. Quartics with three real, distinct double points.

12. The dps being at the vertices of the triangle of reference, the general equation of the quartic may be written

$$ay^2z^2 + bz^2x^2 + cx^2y^2 + 2xyz(lx + my + nz) = 0.$$

The general appearance of the curve depends so largely upon the nature and relative position of the tangents *at* the dps and those *from* the dps that it is here proposed to consider the curves in subdivisions according as the tangents at the dps are real and distinct, consecutive, or imaginary; that is, according as the dps are crunodes, cusps, or acnodes,—the curves in these subdivisions being further classified according as the pair of tangents at a crunode are *divided* or *non-divided;* according as an even or an odd number of these pairs (in the case when all are non-divided) enter the triangle of reference; and according to the nature of the *abnodal* tangents.

Pair of tangents will be used only of two that are tangent at the same dp or from the same dp. A pair of tangents will be called *divided* when two lines, each joining two dps, are concurrent with the tangents of the pair and lie respectively in the two pairs of vertical angles formed by the tangents. (For modification of the term see Art. **58**.)

Tangents from a dp will be called *abnodal* tangents.

---

* See Plate II.

i. *Quartics with three real, distinct crunodes.*

13. The necessary and sufficient conditions being

$$\left.\begin{array}{l} l^2 - bc > 0, \\ m^2 - ac > 0, \\ n^2 - ab > 0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

these conditions will be understood in the discussion of tricrunodal * curves.

For classification on the bases referred to (Art. **12**) it is convenient to have for reference the equations, in solved form, of the crunodal and abnodal tangents. For the equation in its general form (Art. **12**) these are as follows:

Crunodal tangents,

$$\left.\begin{array}{l} ay = \left[- n \pm \sqrt{n^2 - ab}\right]x, \\ bz = \left[- l \pm \sqrt{l^2 - bc}\right]y, \\ cx = \left[- m \pm \sqrt{m^2 - ac}\right]z. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

Abnodal tangents,

$$\left.\begin{array}{l} (m^2-ac)y = \left[cn-lm \pm \sqrt{(cn-lm)^2-(l^2-bc)(m^2-ac)}\right]x, \\ (n^2-ab)z = \left[al-mn \pm \sqrt{(al-mn)^2-(m^2-ac)(n^2-ab)}\right]y, \\ (l^2-bc)x = \left[bm-ln \pm \sqrt{(bm-ln)^2-(n^2-ab)(l^2-bc)}\right]z, \end{array}\right\} \quad . \quad (C)$$

Placing

$$al^2 + bm^2 + cn^2 - abc - 2lmn \equiv D, \dagger$$

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* The term *tricrunodal* will be used only when the three crunodes are real and distinct. When corresponding adjectives are needed for curves in which double points are united *tacnodal*, *oscnodal*, etc., will be employed.

† The function D, in tricrunodal quartics, is the discriminant of the conic

$$ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy = 0,$$

from which the unicursal quartic can be obtained by quadric transformation.

equations (C) may be written

$$\left.\begin{aligned}(m^2 - ac)y &= \left[cn - lm \pm \sqrt{cD}\,\right]x,\\ (n^2 - ab)z &= \left[al - mn \pm \sqrt{aD}\right]y,\\ (l^2 - bc)x &= \left[bm - ln \pm \sqrt{bD}\right]z.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (C')$$

14. The following results, readily deducible from equations (B) and (C), are to be noted:

(1) The sign of the coefficient in the equation of a pair of crunodal tangents is governed by the rational term, and therefore the pair of tangents are *non-divided* if, of the three constants a, b, c, the two appearing in the equation have the *same* sign;

(2) If these two constants have *opposite* signs the sign of the coefficient in the equation is governed by the radical and the pair of tangents are *divided;* therefore,

(3) If a, b, c have the *same* sign the three pairs of crunodal tangents are *all non-divided pairs;* but,

(4) If one of these three constants differ in sign from the other two the *two* pairs of crunodal tangents in whose equation this constant appears are *divided,* the other pair are *non-divided;*

(5) A pair of non-divided crunodal tangents will enter the fundamental triangle if the two constants appearing in rational form in their equation have opposite signs (the system of co-ordinates being so chosen that x, y, z are all positive for a point within the triangle), and will not enter it if those constants have the same sign;

(6) From each crunode there are two * tangents, which, when real, cannot be divided;

(7) All abnodal tangents are real if a, b, c, D have the same sign; all imaginary if a, b, c have same sign, D opposite sign; two pairs are real, one pair imaginary, if one of the constants a, b, c differ in sign from the other two and from $D;$ one pair are real, two pairs imaginary, if D and one of the constants a, b, c differ in sign from the other two;

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* An inflexional tangent at a flecnode counts once as a *crunodal* and once as an *abnodal* tangent.

(8) If $D = 0$ the quartic degenerates, the two tangents of each pair of abnodal tangents becoming coincident; that is, they are not proper tangents, but lines leading to a fourth dp.

15. Combining these results in non-contradictory groups, the fundamental forms of tricrunodal quartics are obtained. Since the number of pairs of real or imaginary abnodal tangents is not changed by projection, nor is a pair of *divided* tangents changed into *non-divided* or vice versa, figures capable of being projected into each other will here be regarded as belonging to the same type. This obviates the necessity of special consideration of twenty-eight of the thirty-two equations arising from different combinations of sign among the coefficients; the four remaining equations give rise to five groups or types of tricrunodal quartics.

16. *Type 1.* Quartic with three pairs of non-divided crunodal tangents, of which an odd number of pairs enter the fundamental triangle.

These characteristics require that a, b, c have the same sign, and that one or all of the constants l, m, n have the opposite sign to a, b, c. Writing the equation so that all constants may be considered intrinsically positive, the only forms fulfilling these conditions are

$$ay^2z^2 + bz^2x^2 + cx^2y^2 - 2xyz(lx + my + nz) = 0, \quad . \quad . \quad (1)$$

and the three projections of the same arising from changing the sign of x, y, or z (that is, changing the sign of two of the constants l, m, n).

$$D \equiv al^2 + bm^2 + cn^2 - abc + 2lmn$$

for this case and is essentially positive; hence all abnodal tangents are real.

The curve can always be projected so as to lie entirely in the finite part of the plane, for there is always one line (and therefore more than one) which cuts the curve in four imaginary points; for instance, the line

$$lx + my + nz = 0$$

which meets the quartic where it meets

$$ay^2z^2 + bz^2x^2 + cx^2y^2 = 0\,;$$

since this latter consists of three acnodes, one at each vertex, and since l, m, n must be different from zero when a, b, c have the same

sign (crunodal conditions), the intersections must be imaginary. Similarly every curve belonging to type 2 or type 3 can also be projected entirely into the finite part of the plane.

The *typical* * figure has the additional characteristics that the crunodal tangents all enter the fundamental triangle and the abnodal all lie outside, as distinguished from the typical projections, for which one pair of crunodal and two pairs of abnodal tangents enter the triangle.

These characteristics give the main outlines for Fig. 1_c. The table [Art. 2, II, (1)] indicates the possibility of two real inflexions, which could evidently exist only by means of an undulation or a 'bay' on one loop, using the term 'bay' † to designate that form of indentation  which is shown in non-singular quartics.

An analytical investigation of the nature of the dts shows that all are real when a, b, c have the same sign, but a reproduction of the work is scarcely needed: the foregoing characteristics suffice for an outline sufficiently accurate to justify reliance on the eye for assurance that three dts are non-isolated; the fourth may be isolated or not. (See note C.)

An illustration of type 1 is afforded by

$$y^2z^2 + z^2x^2 + g^2x^2y^2 - 2xyz[(20 - g)x + (20 - g)y + 9z] = 0,$$

which in the form

$$20xyz(2y + z + 2x) = (yz + zx + gxy)^2$$

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* In the discussion of each type an equation is given which, as given and in all its projections, represents the type; the equation *as given* will be called the *typical equation* and its locus the *typical figure* (shown in Plate II); the projections mentioned in the text (changing sign of two of the constants l, m, n) will be called 'typical projections,' a few figures being given in a separate plate, P.

The typical figure cuts infinity the least possible number of times; it is not difficult to see that the combination of signs chosen in the typical equation is the proper one for this purpose. It is not implied, however, that the locus of a typical equation cannot have more real points at infinity, but simply that, *if* it have, it can be brought into the typical form by a projection that leaves *sign* of D and *all signs* in the typical equation unchanged.

† This term, together with three others—*embayed, in-cusped, out-cusped*—used farther on, are adopted from Mr. H. W. Richmond's paper, 'Cuspidal Quartics," *Quar. Jour. Math.*, vol. XXVI, Feb. 1892.

reveals the dt

$$2y + z + 2x = 0.$$

If $10 > g > 8$ this dt has real, distinct contacts on the 'embayed' loop (Fig. 1_a); if $g = 8$ there is an undulation at the point $x = z = -\frac{y}{4}$ (Fig. 1_b); if $8 > g > 0$ the dt in question is isolated; if $g = 0$ the curve degenerates. (If $g = 10$ the curve is of type 9.)

These three forms (two real, distinct inflexions, two consecutive inflexions, and no real inflexion) will here be regarded only as minor variations of one type.

'Typical projections' are made by sending to infinity a line cutting the fundamental triangle (Plate P, Figs. 1′, 1″, with or without the bay on loop).

17. *Type 2.* Quartic with three pairs of non-divided crunodal tangents, of which an even number of pairs enter the fundamental triangle; all abnodal tangents real.

The typical equation is

$$ay^2z^2 + bz^2x^2 + cx^2y^2 + 2xyz(lx + my + nz) = 0, \quad . \quad . \quad (2)$$

in which all constants are positive.

$$D \equiv al^2 + bm^2 + cn^2 - abc - 2lmn$$

is not necessarily essentially positive for equation (2), hence the condition $D > 0$ is to be added, for the real abnodal tangents.

An odd number of pairs of abnodal tangents enter the triangle of reference.

This, perhaps less evident than preceding results, is seen as follows: Whether the equation be (2) or one of its typical projections, the quantities a, b, c, D, and the product lmn are all positive; from the crunodal conditions we have $l^2m^2n^2 > a^2b^2c^2$, $\therefore lmn > abc$, $\therefore (lmn - abc)D > 0$.

$$\begin{aligned}(lmn - abc)D &\equiv (lmn - abc)(al^2 + bm^2 + cn^2 - abc - 2lmn)\\ &\equiv (cn - lm)(al - mn)(bm - ln) - (l^2 - bc)(m^2 - ac)(n^2 - ab),\end{aligned}$$

which is therefore > 0; hence

$$(cn - lm)(al - mn)(bm - ln) > 0;$$

hence one or three of these binomials must be positive; that is, one pair or all of the abnodal tangents enter the fundamental triangle [equations (C)]. For the typical curve only one pair enter. For, since one of the binomials

must be positive, let $cn - lm > 0$; $\therefore$ $cmn - lm^2 > 0$, $\therefore$ $cmn - lac > 0$; $\therefore$ $mn - al > 0$, $\therefore$ $al - mn < 0$, and similarly $bm - nl < 0$. In the same way, if either $al - mn$ or $bm - ln$ be positive, l, m, n being positive, the other two binomials will be positive.

It is also easily seen that, for the typical curve, the 'even number' of crunodal pairs entering the triangle of reference is zero.

These characteristics suffice for outlining the typical curve (Fig. 2). The seeking of real inflexions by direct analysis is unnecessary; when the preceding conditions are all fulfilled it is perfectly evident that two real inflexions are indispensable, one in the neighborhood of each of the 'loop crunodes'; whether the inflexion be at the crunode or on one or the other side of it depends upon numerical conditions of minor importance.

The four dts are manifestly all real, non-isolated.

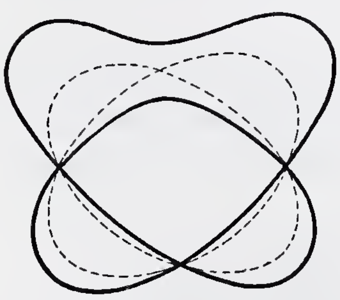
For a numerical example of type 2, see Art. **18**.

Typical projections, for which two pairs of crunodal and respectively three pairs and one pair of abnodal tangents enter the fundamental triangle, are shown in Plate P, Figs. 2′, 2″.

18. *Type 3*. Quartic with three pairs of non-divided crunodal tangents, of which an even number of pairs enter the fundamental triangle; all abnodal tangents imaginary.

The typical equation is equation (2) (Art. **17**), with the condition $D < 0$.

For the typical curve the 'even number' of crunodal tangents entering the triangle of reference is zero, as in type 2. The locus is readily traced to Fig. 3_c, with the question of a possible bay on one outer convexity to be considered.



Three dts are evidently non-isolated; the fourth, isolated or not.

If $|D|$ be small this dt is non-isolated, the locus differing little from the two conics which constitute the locus when $D = 0$, and form the transition locus from type 2 to type 3.

An example of types 2 and 3 is afforded by

$$y^2z^2 + z^2x^2 + g^2x^2y^2 + 2xyz\left[\frac{1 + 2g}{2}x + \frac{1 + 2g}{2}y + 2z\right] = 0,$$

which in the form

$$xyz(x + y + 2z) + (yz + zx + gxy)^2 = 0,$$

reveals the bitangent

$$x + y + 2z = 0.$$

If $g > \dfrac{2 + \sqrt{6}}{2}$ the locus is of type 2; if $g = \dfrac{2 + \sqrt{6}}{2}$ the curve degenerates into two conics; if $\dfrac{2 + \sqrt{6}}{2} > g > 2$ the dt in question still has real, distinct contacts, but now they are on a curve of type 3 (Fig. 3_a); if $g = 2$ there is an undulation (Fig. 3_b), if $2 > g > 0$ this dt is isolated (as in Fig. 1_c, which, however, is not drawn for this special example); if $g = 0$ the locus degenerates.

Fig. 3′, Plate P, is a typical projection of type 3, the fourth dt being isolated.

19. The three preceding types exhaust the possibilities so long as all three pairs of crunodal tangents are non-divided. For the case of divided crunodal tangents that one of the three constants a, b, c which must differ in sign from the other two may of course be taken negative, and there is no loss of generality in choosing any one of the three as the odd one: we choose c, thus making xy the vertex at which the crunodal tangents are non-divided.

All unicursal quartics with divided crunodal tangents can therefore be represented by one of the eight forms

$$ay^2z^2 + bz^2x^2 - cx^2y^2 + 2xyz(\pm lx \pm my \pm nz) = 0,$$

six of which are projections of the other two.

All curves represented by these equations differ in one important particular from all those of types 1, 2, 3; viz., it is impossible to project any of them entirely into the finite part of the plane. This is evident when the general equation of a right line

$$Lx + My + Nz = 0$$

is solved with the equation of the quartic and the result tested for nature of the roots. Rejecting those solutions which refer to dps (that is, keeping L, M, N all different from zero), the following combinations of the four roots will be found impossible, whatever be the sign and numerical values of l, m, n, D: four imaginary; two imaginary and two coincident; two pairs of coincident; four coincident. Hence, *infinity* cuts *the curve in at least two points,*

and therefore the curve has at least two asymptotes; a real dt is impossible, and hence (table, Art. 2) *two real inflexions must exist, there being no isolated dt.* The merest outline of the curve reveals these characteristics unmistakably.

$Lx + My + Nz = 0$. Substitute the value of y from this equation in

$$ay^2z^2 + bz^2x^2 - cx^2y^2 + 2xyz(lx + my + nz) = 0,$$

and impose conditions that the resulting equation may be the same as

$$[z - (\alpha + i\beta)x]\,[z - (\alpha - i\beta)x]\,[z - (\gamma + i\delta)x]\,[z - (\gamma - i\delta)] = 0.$$

The equations of condition are

$$\alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 = \frac{-cL^2}{aM^2},$$

$$2(\alpha + \gamma) = \frac{-2aLM - nMN + mN^2}{aN^2},$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 4\alpha\gamma = \frac{aL^2 + bM^2 - cN^2 - mLN + nLM + lMN - mLN}{aN^2},$$

$$2[\alpha\gamma(\alpha + \gamma) + \alpha\delta^2 + \gamma\beta^2] = \frac{2cLN - lLM + mL^2}{aN^2}.$$

The first of these conditions shows the impossibility of four imaginary roots, since the left side is positive and the right negative. Making $\beta = 0$ to try for two imaginary and two real roots, the same equation of condition shows it is impossible unless α^2 be replaced by $\alpha\alpha'$, in which case $\alpha > 0$, $\alpha' < 0$ may be found; that is, when only two roots are real they must be different. Making all four real, $\beta = \delta = 0$, leaves

$$\alpha^2x^2 = \frac{-cL^2}{aM^2},$$

which is not possible except by replacing α^2 by $\alpha\alpha'$ and making $\alpha\alpha' < 0$, or by replacing γ^2 by $\gamma\gamma'$ and making $\gamma\gamma' < 0$; that is, if all four intersections are real, at least two of them must be distinct.

These results are independent of sign and numerical values of l, m, n, D, and are therefore applicable to all curves represented by

$$ay^2z^2 + bz^2x^2 - cx^2y^2 + 2xyz(\pm lx \pm my \pm nz) = 0.$$

20. Zeuthen (*loc. cit.*, pp. 425–6) notes the existence of quartics that cannot be projected entirely into the finite; examples of the kind in the sextic can be found in Clebsch (*Vorlesungen über Geometrie*); and even earlier than Zeuthen, Cayley (*Collected Papers*, vol. v, No. 361; *Phil. Mag.*, 1865) gives the theorem that a

non-singular quartic can be projected into the finite, and remarks that this is not true for all singular quartics nor for all non-singular sextics. If the quartic be not unicursal it is possible for it to have this characteristic by reason of its consisting of two odd circuits [see Arts. **75** (type 7), **78** (type 8)], but in the unicursal quartic we have the peculiarity that a *single circuit of even order* may be incapable of being projected entirely into the finite. Such a circuit will here be called a *double-odd circuit*. The term is open to the objection that it *suggests two odd circuits*, although used to designate a single even circuit (traceable by one stroke if we could actually pass through infinity). 'Double-odd' is suggested by the fact that the two partial circuits which in practice are drawn for such a circuit have some of the features of an odd circuit.

21. *Type 4.* Quartic with two pairs of divided and one pair of non-divided crunodal tangents; two pairs of real and one pair of imaginary abnodal tangents.

A typical equation is

$$ay^2z^2 + bz^2x^2 - cx^2y^2 - 2xyz(lx + my + nz) = 0 \quad . \quad . \quad (3)$$

(in which all constants are intrinsically positive), with the condition

$$D \equiv al^2 + bm^2 - cn^2 + abc + 2lmn > 0.$$

This equation differs from equation (1) only in the sign of c, and the curve might easily be traced by varying Fig. 1 slightly. Writing the equation in the form

$$S - 2x^2y^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where S is the left member of equation (1), it is seen that the locus lies outside of the S-locus, and for c small differs little from it until it reaches a portion of the plane where x and y are large. The transition from Fig. 1 to Fig. 4_a is through the quartic composed of $z = 0$ and a crunodal cubic. It is easily apparent that an infinity of lines can be found which cut the curve in but two real points, so that it is always possible to retain all the dps in the finite part of the plane and at the same time have but two real points at infinity.

The conditions necessary in order that the curve may have the specified characteristics are also fulfilled by

$$ay^2z^2 + bz^2x^2 - cx^2y^2 + 2xyz(lx + my + nz) = 0, \quad . \quad . \quad (4)$$

if $D > 0$. The locus is the same in its main features as the locus of (3), the difference lying mainly in the position of inflexions.

The real inflexions have a wide range, but are excluded from the loop and the arcs joining the loop to the other crunodes, as is evident in various ways. Figs. 4_b, 4_c show two positions of inflexions essentially different from those in Fig. 4_a. Whether an inflexion lie in one or the other portion of a triangle which is cut by infinity is of course only a matter of projection.

For the locus of equation (3) the non-divided crunodal tangents enter the fundamental triangle, the abnodal do not. Typical projections are shown (Plate P, Figs. $4'$, $4''$) for the two cases of non-divided crunodal and all abnodal tangents entering the triangle (Figs. $4_a'$, $4_b'$), and the non-divided crunodal and one pair of abnodal lying outside of the triangle (Figs. $4_a''$, $4_b''$). It is to be noted that these last figures ($4_a''$, $4_b''$) are typical figures, as well as typical projections, being the first typical projections which do not cut infinity oftener than the original typical figure.

$$y^2z^2 + z^2x^2 - x^2y^2 - 2xyz(x + y + \tfrac{3}{2}z) = 0$$

(Fig. 4_a) is a simple example of type 4.

22. *Type 5.* Quartic with two pairs of divided and one pair of non-divided crunodal tangents; one pair of real and two pairs of imaginary abnodal tangents.

The necessary conditions here, as in the preceding case, are satisfied, so far as signs in the equation are concerned, by any one of the eight forms of the equation given in Art. **19**, the distinguishing condition between this and type 4 being in the sign of D, which is negative for the present type. As typical equation take

$$ay^2z^2 + bz^2x^2 - cx^2y^2 + 2xyz(lx + my + nz) = 0 \quad . \quad . \quad (4)$$

with the condition $D < 0$.

For this form the non-divided crunodal and the real abnodal tangents all lie outside of the fundamental triangle. As in type 4, more than two real points at infinity can be avoided. The curve could be traced by the help of Fig. 3, just as Fig. 1 was used for type 4. Equations (3) and (4) cannot be projected into each other, yet each represents both types 4 and 5. The difference, as already remarked, is chiefly in the position of inflexions, which is not here made a basis of classification.

Fig. 5_a, the locus (approximate) of

$$y^2z^2 + z^2x^2 - x^2y^2 + 2xyz(\tfrac{3}{4}x + \tfrac{3}{4}y + \tfrac{13}{5}z) = 0,$$

is a typical figure: the real inflexions are excluded from the two short arcs joining the crunodes yz, zx in Fig. 5 and from the corresponding portions in all projections and variations. Fig. 5_b (a projection of Fig. 5_a) and Fig. 5_c show some of the variations in position of inflexions. Figs. $5_a'$, $5_c'$, $5_c''$, Plate P, are typical projections.

One important particular in which this type differs from the preceding is perhaps worthy of mention; viz., the line joining the two crunodes with divided tangents being projected to infinity, the coefficients can be so specialized in type 5 that the remaining crunode becomes a centre for the curve, all that is necessary being $l = m = 0$.

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23. In order that a cusp may occur at any vertex the crunodal condition [Art. **13**, (A)] for that vertex must be replaced by a cuspidal condition; that is, the sign of inequality must be replaced by that of equality. Whenever this change is consistent with all other analytical conditions for the existence of a given type of tricrunodal quartic there is a quartic not differing essentially from the tricrunodal, except in having the cusp instead of crunode at the vertex in question. From substituting the cuspidal condition at the dps of the different tricrunodal types, contradictions arise except for the crunodes at which the figure presents a loop, as was to be expected from the geometrical conception of a cusp as an evanescent loop.

That a crunode at which the tangents are divided must pass into the opposite kind before these tangents can become the consecutive tangents at a cusp is evident, since a side of the triangle cannot become a cuspidal tangent. If in type 5 a cuspidal condition be imposed at the vertex at which the tangents are *non-divided*, D becomes a perfect square, whereas $D < 0$ is an existence condition for type 5. A similar thing occurs if any cuspidal condition be imposed for type 3. For type 2, suppose the 'non-loop-crunode' is at xy: then $cn - lm > 0$, $al - mn < 0$, $bm - ln < 0$; these with the cuspidal condition $n^2 - ab = 0$ give $a^2b^2c^2 > l^2m^2n^2$, whereas $m^2 - ac > 0$, $l^2 - bc > 0$, $n^2 - ab = 0$ give $a^2b^2c^2 < l^2m^2n^2$.

Substituting the cuspidal for the crunodal condition at each 'loop-crunode,' it is found that the one loop of type 4, one or both loops of type 2, and one, two, or all three of type 1 can be made evanescent.

When a loop becomes evanescent one of the tangents from each of the other dps is absorbed by a side of the fundamental triangle. The evanescence of the first loop causes the absorption of two dts by the tangents from the new cusp (that is, the tangents from the crunode that undergoes change), real abnodal tangents absorbing real dts and imaginary absorbing imaginary; at the evanescence of the second loop one real dt is thus absorbed (the tricrunodal quartics with imaginary bitangents have no second loop); when the third loop becomes evanescent the one real dt is unabsorbed and is isolated.

The characteristics of each type in respect of number and position of real abnodal tangents having been given in full for tricrunodal quartics, will for the cuspidal forms be indicated simply by specifying the crunodal form from which each is derived.

ii. *Quartics with two real, distinct crunodes and one cusp.*

Equations (1), (2), (3), (4) with $D > 0$.*

24. *Type 6.* Limit of type 1 when one loop becomes evanescent. One of the dts may be either non-isolated (Fig. 6_a) or isolated (Fig. 6_b).

Type 7. Limit of type 2 when one loop becomes evanescent. Both dts are non-isolated. $I = 2$.†

Type 8. Limit of type 4 when the loop becomes evanescent. The curve is still a double-odd circuit, the two dts are imaginary, and $I = 2$. Fig. 8 is typical, but wide variation in position of real inflexions is possible, as in type 4.

iii. *Quartics with one crunode and two real, distinct cusps.*

Equations (1), (2), with $D > 0$.

25. *Type 9.* Limit of type 1 when two of the loops become

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*From this point, of course, the conditions (A), (Art. **13**), must be modified for each division, according to the combination of crunodes, cusps, and acnodes desired.

† As in the table (Art. **2**), I designates the number of real inflexions.

evanescent. The dt is either non-isolated (Fig. 9_a) or isolated (Fig. 9_b).

Type 10. Limit of type 2 when both loops become evanescent. The dt is non-isolated. $I = 2$.

iv. *Quartics with three real, distinct cusps.*

Equation (1).

26. *Type 11* $\equiv$ iv. Limit of type 1 when all loops become evanescent. The one dt is isolated. $I = 0$.

It seems scarcely necessary to show typical projections of types 6–11.

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27. Pursuing the investigation, it is found that in every case in which the crunodal condition could be replaced by the cuspidal the latter may be legitimately replaced by an acnodal condition,* one, two, or three acnodes being simultaneously possible for the figures that allowed as many cusps. In the first stages the acnodal curve arising from a slight variation of the cuspidal form will lie close to the latter in the neighborhood of the cusp, and, bending to fit the two convexities at the cusp, will have two new real inflexions. It is evident from the figures themselves, drawn from the cuspidal forms, and the equations of the dts would show, that the dts absorbed by the cusp and again set free in the acnodal curve are set free *real* or *imaginary*, according as they were real or imaginary in the tricrunodal figure. Whatever be the nature of the other dps of the quartic, when real dts are set free by the change from cusp to acnode they pass in between the acnode and that portion of the circuit that has just lost its cusp, and are *non-isolated.* Whether they may eventually become isolated or not depends upon other features of the curve, but they continue to pass in between the acnode and the circuit as before.

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* Where the attempt to impose a cuspidal condition results in degeneration the acnodal condition may be imposed, and a proper curve results; but the sign of D is changed, so that the proper curve is strictly to be regarded as coming from another of the crunodal forms.

v. *Quartics with two real, distinct crunodes and one acnode.*

Equations (1), (2), (3), (4) with $D > 0$.

28. Equations (C), (Art. **13**), show that two pairs of abnodal tangents must be real and *divided* for these curves, while the third pair may be either real and non-divided or imaginary.

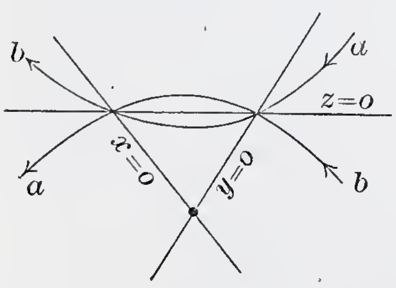
Type 12. Quartic with real, non-divided tangents from the acnode, real divided tangents from each of the two crunodes; crunodal tangents non-divided.

These curves are derived from type 6 (originally 1) and type 7 (originally 2) by replacing the cusp by an acnode. Three dts are non-isolated; the fourth is non-isolated in the incipient stages of the curves (Figs. 12_a, 12_c) derived from cuspidal curves (6_a, 7) which have no isolated dt, but by gradual shrinking of the loop the bay may disappear, the fourth dt becoming isolated (Figs. 12_b, 11_d). $I \not< 2$.

In their early stages the figures derived from type 6 and those from type 7 look quite dissimilar, it is true, but they are nevertheless of the same type when the crunodal and abnodal tangents are made the basis of subdivision: for by projection and gradual variation of constants Figs. 12_c, 12_d can pass into Figs. 12_a, 12_b without degeneration of the curve or change of type-characteristics.

Type 13. Quartic with imaginary tangents from the acnode, real divided tangents from each of the two crunodes; crunodal tangents divided.

This curve is derived from type 8 (originally 4). It is a double-odd circuit; all dts are imaginary, as in types 4, 8. There being no isolated dt, $I = 4$ necessarily. (Table, Art. **2.**) The form which apparently might exist with $I = 0$ is to be guarded against; the branch a returns from infinity with its convexity properly directed so far as $y = 0$ and $z = 0$ are concerned, but not so with respect to its asymptote—and it must have an asymptote, since it *cuts* infinity; the same is true of branch b.



This point is emphasized because this is the only figure with two real distinct crunodes and an acnode in which, at a cursory glance, it might seem that $I = 0$ were possible, —a case provided for by the table [Art. **2**, II, (5)] but non-existent when all dps are real and distinct. (See Art. **4.**)

There being four real inflexions, there is a large scope for variation in appearance. The various forms shown and suggested for the original tricrunodal figure will suggest some of the many possible forms here. The acnodal figure derived directly from Fig. $4_b''$ is scarcely recognizable at first glance as being of the same type as Fig. 13. It looks much like a derivation from Fig. 5, and can be so obtained (see foot-note to Art. **27**) but need not be so considered. Much as the inflexions may vary in position, however, there are limitations. For instance, if one of the crunodes be a *bi*-flecnode the other must either not be a flecnode at all or else be also a *bi*-flecnode. This appears readily enough from the equation as given; the reason is also apparent when we consider how a biflecnode arises. (See Art. **70.**)

vi. *Quartics with one crunode, one cusp, and one acnode.*

Equations (1), (2) with $D > 0$.

29. *Type 14* $\equiv$ vi. Derivative from type 9 (originally 1) and type 10 (originally 2) when one cusp is replaced by an acnode. One of the two dts is non-isolated, the other non-isolated (Figs. 14_a, 14_c) or isolated (Figs. 14_b, 14_d). $I \nless 2$, although the table [Art. **2**, II, (18)] provides for $I = 0$. (See also remarks under type 12.)

vii. *Quartics with two real, distinct cusps and one acnode.*

Equation (1).

30. *Type 15* $\equiv$ vii. Derivative from type 11 (originally 1) when one cusp is replaced by an acnode. The dt is isolated. $I = 2$.

viii. *Quartics with one crunode and two acnodes.*

Equations (1), (2) with $D > 0$.

31. *Type 16* $\equiv$ viii. Derivative from types 9, 10 (originally 1, 2) when both cusps are replaced by acnodes.

The abnodal tangents are all real, two pairs divided, one pair (from the crunode) non-divided. Two dts are non-isolated; the other two may be both non-isolated (Figs. 16_a, 16_e), one isolated (Figs. 16_b, 16_c, 16_f), or both isolated (Figs. 16_d, 16_g). $I \nless 2$, although possible according to the table [Art. **2**, II, (9). See also Art. **4**]. (See also remarks under type 12.)

One form that might present itself to the mind as a possible

special form of type 16 is to be guarded against; viz., a form having a biflecnode together with one 'bay dt' and one isolated dt. Figs. 16_a, 16_d, 16_e, 16_g could assume the special forms arising from locating at the crunode the two real inflexions not situated in a bay, but not so Figs. 16_b, 16_c, 16_f: these may become flecnodal, but not *bi*-flecnodal. (See Art. **70.**)

ix. *Quartics with one cusp and two acnodes.*

Equation (1).

32. *Type 17* ≡ ix. Derivative from type 11 (originally 1) when two of the cusps are replaced by acnodes. One dt is isolated; the other is non-isolated (Fig. 17_a) or isolated (Fig. 17_b).

x. *Quartics with three acnodes.*

Equation (1).

33. *Type 18* ≡ x. Derivative from type 11 (originally 1) when all the cusps are replaced by acnodes. All three pairs of abnodal tangents are real and non-divided. *One dt is isolated;* the others may be all *three* non-isolated (Fig. 18_a), one isolated (Fig. 18_b), two isolated (Fig. 18_c), or all three isolated (Fig. 18_d); that is, Figs. 18_a, 18_b, 18_c, 18_d have respectively one, two, three, four isolated dts.

If $l = m = n = 0$ equation (1) represents three acnodes without a real circuit.

34. Noticing the position of those dts that become isolated in the preceding figures, and bearing in mind that an isolated dt cannot pass through a dp and so cannot enter a triangle which it did not originally enter, the following is to be remarked concerning positions of isolated dts in the preceding unicursal quartics: The one isolated dt of Figs. 1_c, 6_b, 9_b, 11, 12_b, 14_b, 15, 16_b, 17_a, 18_a, and *that one* of Figs 16_d, 17_b, 18_b, 18_c, 18_e which originated in the isolated dt of Fig. 1_c, lie outside of the fundamental triangle; the second isolated dt of Figs. 16_d, 17_b, 18_c, the second and third of Fig. 18_c, the second, third, and fourth of Fig. 18_d, and the *one* of Fig. 16_c cut the triangle, each separating two acnodes from the circuit; the one isolated dt of Figs. 12_d, 14_d, 16_f, and both those of Fig. 16_g cut the triangle, each separating one acnode from a loop. Of course, by projection, an acnode may take quite a different position in relation to the curve, and a different triangle becomes

the fundamental triangle, but for future reference it was perhaps worth while to note the positions of the isolated dts for the figures as given, whence they can be located for any projection.

§ II. Quartics with one distinct and two consecutive double points.

A. *Quartics with a tacnode and a distinct double point.*

35. Tacnodes formed by the contact of real branches are here distinguished as *embrassement* , in which both branches lie on the same side of the tangent, and *opposition* , in which the branches lie on opposite sides of the tangent.* When the branches are imaginary the point is called an *isolated tacnode.*

A_1. *Quartics with an embrassement and a distinct double point.*

36. By locating the tacnode at xy and taking the tangent there for $y = 0$, any quartic with an *embrassement* may be represented by

$$(yz - mx^2)(yz - m'x^2) = 2cxy^2z + fx^3y + gx^2y^2 + dy^3z + hxy^3 + ly^4,$$

where m, m' are intrinsically positive and $m \neq m'$.

If a third dp be at xz, $d = h = l = 0$. By choosing $z = 0$ so that the tangents at xz are

$$z^2 - gx^2 = 0$$

c vanishes and the equation becomes

$$\left.\begin{array}{c}(yz - mx^2)(yz - m'x^2) = fx^3y + gx^2y^2, \\ m \neq m'.\end{array}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

The tangents from xy and xz are

$$(m - m')^2x^2 + 4fxy + 4gy^2 = 0,$$
$$(f^2 + 4mm'g)x^2 + 2f(m + m')xz + (m - m')^2z^2 = 0,$$

which are all real if

$$f^2 - g(m - m')^2 > 0,$$

* Cramer (*Analyse des Lignes Courbes*, 1750) calls this latter variety *osculation*, but this term is perhaps open to objections owing to its ordinary use as indicating contact of an order higher than the first.

all imaginary if

$$f^2 - g(m - m')^2 < 0.$$

This function plays the rôle of D in § I.

The point xz is a crunode if $g > 0$, a cusp if $g = 0$, an acnode if $g < 0$. The sign of f is immaterial, being a mere matter of projection. The simplest method of tracing is to use the two conics $yz - mx^2 = 0$, $yz - m'x^2 = 0$, the right lines $y = 0$, $fx + gy = 0$, and the abnodal tangents (when real), as excluding curves.

xi. *Quartics with an embrassement and a crunode.*

$$g > 0.$$

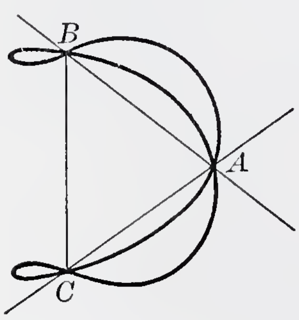
37. *Type 19.* Quartic with real tangents from the *embrassement* and from the crunode.

$$f^2 - g(m - m')^2 > 0.$$

It has been remarked (Art. **4**) that parts of the table (Art. **2**) may possibly be inapplicable to cases in which dps are consecutive or coincident. Each case, therefore, requires examination with respect to this point. The Plückerian characteristics obtain in all cases, however, and are of service in testing whether T and I agree with the table.

The tangent at a tacnode counts as two non-isolated dts; hence type 19 has but two others, which are plainly in evidence in Fig. 19; a bay, yielding two additional real inflexions, is thus impossible. $I = 2$.

The curve may be regarded as the limit of type 2 when the point A moves up to a position consecutive to one of the 'loop-crunodes'; for instance, AB swings on B as a pivot to a position consecutive to BC, and AC swings on C as a pivot until point A is consecutive to C.



Type 20. Quartic with imaginary tangents from the *embrassement* and the crunode.

$$f^2 - g(m - m')^2 < 0.$$

If $| f^2 - g(m - m')^2 |$ be small the curve differs little from the two conics which constitute the transition curve from type 19 to type 20, and there is a bay (Fig. 20_a); all four dts are thus non-

isolated. As $| f^2 - g(m - m')^2 |$ increases the fourth dt may become isolated (Fig. 20_b).

The type is the limit of type 3, just as type 19 is of type 2.

xii. *Quartics with an embrassement and a cusp.*

$$g = 0.$$

38. *Type 21* $\equiv$ xii. Limit of type 19 when the crunodal loop becomes evanescent, or limit of type 7 when varied as was type 2 to give type 19. The two possible dts being at the *embrassement*, an extra bay is not possible. $I = 2$.

xiii. *Quartics with an embrassement and an acnode.*

$$g < 0.$$

39. *Type 22* $\equiv$ xiii. Derivative from type 21 when the cusp is replaced by an acnode. (See Art. **27**.)

Writing the equation in the form

$$S \equiv (yz - mx^2)(yz - m'x^2) - fx^3y = gx^2y^2,$$

the relation to the cuspidal figure

$$S = 0$$

becomes apparent.

Three dts are non-isolated; if $| g |$ be small the fourth also is non-isolated (Fig. 22_a); with sufficient increase of $| g |$ this dt becomes isolated (Fig. 22_b). (The symmetry of Fig. 22_b would of course not result from increase of $| g |$, but from $f = 0$.) $I \not< 2$.

This is the limit of type 12 when the crunodes become consecutive.

A_2. *Quartics with an opposition and a distinct double point.*

40. Making the same choice of co-ordinates as for the *embrassement* (Art. **36**), any quartic with an *opposition* may be represented by

$$(yz - mx^2)(yz + m'x^2) = 2cxy^2z + fx^3y + gx^2y^2 + dy^3z + hxy^3 + ly^4,$$

where m, m' are intrinsically positive and $m = m'$ is not excluded: and any quartic with an *opposition* and another dp may be represented by

$$(yz - mx^2)(yz + m'x^2) = fx^3y + gx^2y^2. \quad . \quad . \quad . \quad (6)$$

The abnodal tangents are

$$(m+m')^2x^2+4fxy+4gy^2=0,$$

$$(f^2-4mm'g)x^2+2f(m-m')xz+(m+m')^2z^2=0;$$

the former are real if

$$f^2-g(m+m')^2>0,$$

in which case the latter are imaginary, and vice versa.

Using the excluding curves corresponding to those suggested for the *embrassement* (they are the same except that the sign of m' is changed), the main features in the form of the locus are readily discovered.

xiv. *Quartics with an opposition and a crunode.*
$g>0$.

41. *Type 23.* Quartic with real tangents from the *opposition*, imaginary from the crunode.

$$f^2-g(m+m')^2>0.$$

The locus is a double-odd circuit, the limit of type 4 when the two crunodes with divided tangents become consecutive. The relation is most easily seen by comparing Fig. 23 with Fig. $4_b''$ (Plate P). Two of the imaginary dts of type 4 have become the real tangents at the tacnode; the other two remain imaginary. The two real inflexions have the range of branch (a), but are confined to that branch, the tangents from the tacnode being non-divided.

Type 24. Quartic with imaginary tangents from the *opposition*, real from the crunode.

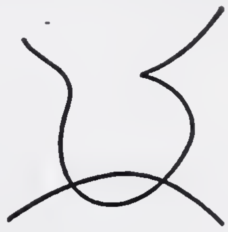
$$f^2-g(m+m')^2<0.$$

The locus is again a double-odd circuit, the limit of type 5 when the crunodes with divided tangents become consecutive, Figs. 24_a, 24_b, 24_c being respectively the limits of Figs. 5_a, 5_b, 5_c.

Two dts are imaginary. $I=2$. If we apply the term 'half-circuit' to that portion of the curve which is traced by starting at the tacnode, passing once through infinity, and returning to the tacnode, the two real inflexions are situated one on each half-circuit, whereas in type 23 both are on one half-circuit.

xv. *Quartics with an opposition and a cusp.*

$g = 0.$

42. *Type 25* $\equiv$ xv. Limit of type 23 when the loop becomes evanescent (or of type 8 when, after projection into , the crunodes become consecutive).

The two dts at the tacnode are the only ones; the two real inflexions have the range of the half-circuit not containing the cusp.

xvi. *Quartics with an opposition and an acnode.*

$g < 0.$

43. *Type 26* $\equiv$ xvi. Derivative from type 2 when the cusp is replaced by an acnode.

The tangents from the tacnode are real and separated by $x = 0$, so that two real inflexions exist on *each* half-circuit; the two dts other than those at the tacnode being imaginary, the real inflexions cannot pass off.

The inflexions have such a wide range, each pair in the one half-circuit, that the curve in some of its forms might hardly be recognized as belonging to the same type as Fig. 26, but by projection it can always be brought into recognizable shape.

The type is the limit of type 13, as type 25 was of type 8.

The Conchoid of Nicomedes is a familiar curve belonging to types 23, 25, 26 according as it has a crunode, cusp, or acnode. If Figs. 23, 25, 26 be projected by sending to infinity a line through the tacnode and passing between the tangents from the tacnode, the features of the Conchoid become apparent.

A_3. *Quartics with an isolated tacnode and a distinct double point.*

44. So far it has been assumed that m, m' in equations (5), (6) are real. If, however, in equation (5) we have

$$\begin{aligned} m &= \alpha + i\beta, \\ m' &= \alpha - i\beta, \end{aligned} \qquad \beta \neq 0,$$

the equation may be written

$$y^2z^2 - 2\alpha yzx^2 + (\alpha^2 + \beta^2)x^4 = fx^3y + gx^2y^2, \quad . \quad . \quad (7)$$

in which all coefficients are real. The point xy is thus a point at which the tangents are real and coincident (algebraically), while the next approximation to the curve gives the two imaginary conics

$$yz - (\alpha + i\beta)x^2 = 0,$$
$$yz - (\alpha - i\beta)x^2 = 0.$$

The point is, therefore, *isolated*, but by reason of its real tangents must not be regarded as an ordinary isolated point (acnode), but as an *isolated tacnode*.

The tangents from xy and xz are

$$gy^2 + fxy - \beta^2x^2 = 0,$$
$$[f^2 + 4g(\alpha^2 + \beta^2)]x^2 + 4\alpha fxz - 4\beta^2z^2 = 0,$$

which are real if $f^2 + 4g\beta^2 > 0$; if $f^2 + 4g\beta^2 < 0$ the circuit is imaginary.

Equation (7) written in the form

$$(yz - \alpha x^2)^2 = x^2(gy^2 + fxy - \beta^2x^2)$$

shows that the contacts of tangents from the isolated tacnode are on the conic

$$yz - \alpha x^2 = 0.$$

Any bays not shown in the figures given as representative will readily be seen to be prevented by the Plückerian characteristics and the fact that here, just as in the case of a tacnode formed by real branches, *the tangent at the tacnode counts as two dts.*

This fact appears as follows: let any right line not passing through one of the dps of the curve be

$$z = lx + my, \qquad l, m \neq 0.$$

Substituting this in equation (7) and imposing conditions that the resulting equation may be

$$[(x - ay)(x - by)]^2 = 0,$$

thus making $z = lx + my$ a dt, the equations of condition are

$$2(a + b)(\alpha^2 + \beta^2) = f + 2l\alpha,$$
$$(a^2 + b^2 + 4ab)(\alpha^2 + \beta^2) = l^2 - 2\alpha m - g,$$
$$ab(a + b)(\alpha^2 + \beta^2) = - lm,$$
$$a^2b^2(\alpha^2 + \beta^2) = m^2;$$

these, solved for l and m, give

$$\begin{cases} l = \dfrac{-f}{2(\alpha - \sqrt{\alpha^2 + \beta^2})}, \\[2ex] m = \dfrac{-g}{2(\alpha - \sqrt{\alpha^2 + \beta^2})}; \end{cases}$$

and

$$\begin{cases} l = \dfrac{-f}{2(\alpha + \sqrt{\alpha^2 + \beta^2})}, \\[2ex] m = \dfrac{-g}{2(\alpha + \sqrt{\alpha^2 + \beta^2})}. \end{cases}$$

These being the *only* sets of values that satisfy the conditions, there are but *two* dts to the curve proper, viz.,

$$z = \frac{-(fx + gy)}{2(\alpha + \sqrt{\alpha^2 + \beta^2})},$$

$$z = \frac{-(fx + gy)}{2(\alpha - \sqrt{\alpha^2 + \beta^2})},$$

when $g \neq 0$, and *none* when $g = 0$, since the preceding equations are the equations of abnodal tangents when $g = 0$. The distinction of real and imaginary not having been made with regard to l, m, imaginary dts would have been found had they existed; hence the other two dts must be the tangent at the isolated tacnode counted twice.

These outlines are practically sufficient to show the general form of types 27–29.

xvii. *Quartics with an isolated tacnode and a crunode.*

$g > 0$.

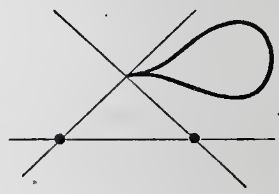
45. *Type 27* ≡ xvii. Fig. 27_a is typical. Two real inflexions are unavoidable: they may be one on each loop; both on one loop; one at the crunode; or (when $\alpha = f = 0$) both at the crunode. Symmetry to $x = 0$, $z = 0$, or to both, is possible.

The type may be regarded as the limit of those forms of type 16 which have two isolated dts, when the acnodes become consecutive; for instance, Fig. 27_a comes directly from Fig. 16_g; the two isolated dts of Fig. 16_g, each of which separates an acnode from the corresponding loop and the acnodes from each other (Art. **34**), become consecutive with the side of the triangle and become con-

secutive, *non-isolated* dts, their real contacts being at these two consecutive acnodes which form the isolated tacnode. Similarly, Fig. 27_c is the limit of Fig. 16_d, the figure being first projected so that the real circuit lies outside of the fundamental triangle.

xviii. *Quartics with an isolated tacnode and a cusp.*

$g = 0.$

46. *Type 28* $\equiv$ xviii. Limit of type 27 when a loop becomes evanescent, or limit of type 17 (Fig. 17_b), projected into , when the acnodes become consecutive. $I = 2$.

xix. *Quartics with an isolated tacnode and an acnode.*

$g < 0.$

47. *Type 29* $\equiv$ xix. Derivative from type 28 when the cusp is replaced by an acnode.

The two dts to the curve proper are non-isolated (Fig. 29_a) for $|g|$ small; as $|g|$ increases one (Fig. 29_b), or both (Fig. 29_c), become isolated.

These three figures are limits of Figs. 18_b, 18_c, 18_d, when two acnodes become consecutive. Bearing in mind the location of the isolated dts, we can determine which are the two acnodes in Fig. 18_b that may be made consecutive and which the two in Fig. 18_c that cannot be.

If $f^2 + 4g\beta^2 = 0$ the locus of equation (7) is the real conic $yz - \alpha x^2 = 0$ together with two real lines.

If $f^2 + 4g\beta^2 < 0$ equation (7) represents simply the isolated tacnode xy and the acnode at xz.

48. Ranking the isolated tacnode as a veritable tacnode, not as two distinct acnodes, and therefore placing types 27–29 respectively under (1), (14), and (3)–(5), II, *not* under (8), (20), and (11)–(13), II, we find that unicursal tacnodal quartics agree with the table (Art. **2**); but it is to be noted that there is no tacnodal quartic corresponding to (15), II and none corresponding to (5), II until a type (Fig. 29_e) is reached in which the *circuit* is non-singular.

B. *Quartics with a node-cusp and a distinct double point.*

49. If in the general equation of the quartic with an *embrassement* at xy (Art. **36**)

$$m' = m,$$

the equation becomes

$$(yz - mx^2)^2 = 2cx^2yz + fx^3y + gx^2y^2 + hxy^3 + ly^4 + dy^3z$$

and the two expansions at xy no longer separate at x^2. If

$$f + 2cm \neq 0$$

the expansions are

$$y = mx^2 \pm nx^{\frac{5}{2}},$$

where $n \equiv \sqrt{m(f+2cm)} \neq 0$; and the point xy is a node-cusp (Art. **3**). If another dp be at xz and $z = 0$ be so chosen that $z^2 - gx^2 = 0$ are the tangents, the equation becomes

$$(yz - mx^2)^2 = fx^3y + gx^2y^2, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

which with the condition $f \neq 0$ represents any quartic with a node-cusp and another dp.

xx. *Quartics with a node-cusp and a crunode.*

$g > 0.$

50. *Type 30* $\equiv$ xx. Limit of type 19 when the loop connected with the *embrassement* becomes evanescent, or limit of type 7 when one crunode moves up to the cusp. Both dts are manifestly non-isolated, one being at the node-cusp. $I = 2$, one of the inflexions being involved in the node-cusp.

xxi. *Quartics with a node-cusp and a cusp.*

$g = 0.$

51. *Type 31* $=$ xxi. Limit of type 21 when the loop becomes evanescent, or limit of type 10 when the crunode moves up to one of the cusps. $I = 2.$

xxii. *Quartics with a node-cusp and an acnode.*

$g < 0$.

52. *Type 32* $\equiv$ xxii. Derivative from type 31 when the cusp is replaced by an acnode, or limit of type 22 when a loop becomes evanescent, or limit of type 14 when the crunode moves up to the cusp.

The dt at xy is of course non-isolated; the other is non-isolated if $|g|$ be small (Fig. 32_a), or isolated if $|g|$ be sufficiently large (Fig. 32_b). This dt lies between the acnode and the circuit. $I \nless 2$.

Since a node-cusp involves a real inflexion, the cases (15), (18), (22), II of the table (Art. **2**) cannot be represented among quartics with a node-cusp; (18) not being represented among quartics with distinct dps either, is therefore non-existent.

An isolated node-cusp is impossible, since the expansions are real for $x > 0$ and n real or for $x < 0$ and n imaginary.

The node-cusp is a dp of transition from the *embrassement* to the isolated tacnode: $m \neq m' \neq 0$, *embrassement;* $m = m' \neq 0$, node-cusp; $m = m' = 0$, cubic and right line; m and m' conjugate imaginaries, isolated tacnode.

§ III. Quartics with three consecutive double points.

A. *Quartics with an oscnode.*

53. If in the general equation of the quartic with a node-cusp (Art. **49**)

$$f + 2cm = 0$$

the equation may be written

$$(yz - mx^2 - cxy)^2 = dy^3z + gx^2y^2 + hy^3x + ly^4,$$

in which g is $g - c^2$ of the original equation.

If $md + g \neq 0$ the expansions at xy,

$$y = mx^2 + mx^3(c \pm \sqrt{md + g}),$$

separate at x^3, and the point is an oscnode. A dp at xz being impossible without degeneration, it is convenient to send $z = 0$ to infinity and use the equation in Cartesian co-ordinates,

$$(y - mx^2 - cxy)^2 = dy^3 + gx^2y^2 + hxy^3 + ly^3. \quad . \quad . \quad (9)$$

Although c is placed strongly in evidence it may vanish. The essential conditions for an oscnodal quartic are $m \neq 0$, $md + g \neq 0$, and d, h, l not all zero simultaneously.

The tangents from the oscnode,

$$(d^2 + 4l)y^2 + 4(cd + h)xy + 4(md + g)x^2 = 0,$$

are real if $(cd + h^2) - (d^2 + 4l)(md + g) > 0$.

The expansions at the oscnode are real if $md + g > 0$.

xxiii. *Quartics with an oscnode formed by real branches.*

$$md + g > 0.$$

54. *Type 33.* Quartic with real tangents from the non-isolated oscnode.

$$(cd + h)^2 - (d^2 + 4l)(md + g) > 0.$$

The tangent at the oscnode counting as three dts (Art. **3**), the fourth dt must be real. The expansions at xy show that the branches cross as well as touch. These, with the preceding characteristics, are sufficient to give a figure of which Fig. 33_a may be taken as the type. A simple example is

$$(y - x^2)^2 = y^3(1 - y).$$

The curve touches $x = 0$ if $d^2 + 4l = 0$, cuts it twice if $d^2 + 4l > 0$, presenting forms (e.g., Fig. 33_b) which, even when projected entirely into the finite, look very different from the symmetrical curve taken as type, but whose main features are nevertheless the same.

The type is the limit of type 2 when all three crunodes become consecutive, or the limit of type 19 when the crunode moves up beside the tacnode. The fourth dt is non-isolated. $I = 2$.

Type 34. Quartic with imaginary tangents from the non-isolated oscnode.

$$(cd + h)^2 - (d^2 + 4l)(md + g) < 0.$$

This is the limit of type 3 when the three crunodes become consecutive, or of type 20 when the crunode moves up to the tacnode, varying as in figure (α), not as in figure (β).

The fourth dt may be non-isolated or isolated. A simple example is

$$\left(y - x^2 - \frac{y^2}{2}\right)\left(y - x^2 - \frac{y^2}{3}\right) = Ax^2y^2,*$$

which for $A > \frac{1}{6}$, $A = \frac{1}{6}$, and $\frac{1}{6} > A > 0$ gives respectively Figs. 34_a, 34_b, 34_c.

xxiv. *Quartics with an isolated oscnode.*

$$md + g < 0.$$

55. The tangent at the isolated oscnode counts as three dts, just as in the case of the non-isolated oscnode; the curve proper has but one dt, which is isolated or not. The tangents from the oscnode are real.

Type 35 $\equiv$ xxiv. A simple example is afforded by the illustration given for type 34,

$$\left(y - x^2 - \frac{y^2}{2}\right)\left(y - x^2 - \frac{y^2}{3}\right) = Ax^2y^2,$$

when $A < 0$. For $|A| > \frac{1}{6}$, $|A| = \frac{1}{6}$, and $\frac{1}{6} > |A| > 0$ this gives respectively Figs. 35_a, 35_b, 35_c.

The one distinct dt must lie between the real circuit and the *concavity* of the oscnode; that is, as shown in Fig. 34, not thus  .

For the symmetrical curve this is easily seen as follows: write the equation

$$2m^2x^2 = 2my - gy^2 \pm \sqrt{y^3[(g^2 + 4lm^2)y + 4m(md + g)]},$$

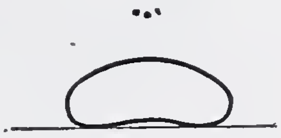
in which the four factors of the radicand represent the dts, the distinct one being

$$(g^2 + 4lm^2)y + 4m(md + g) = 0.$$

The oscnode is isolated and its concavity turned upward $\because$ if $md + g < 0$ and $m > 0$. Suppose, now, $g^2 + 4lm^2 > 0$: then for $y > 0$ and $< \dfrac{-4m(md + g)}{g^2 + 4lm^2}$ (a positive quantity under the supposition), the above radicand is negative and the curve is imaginary: that is, the dt $y = \dfrac{-4m(md + g)}{g^2 + 4lm^2}$ lies between $y = 0$

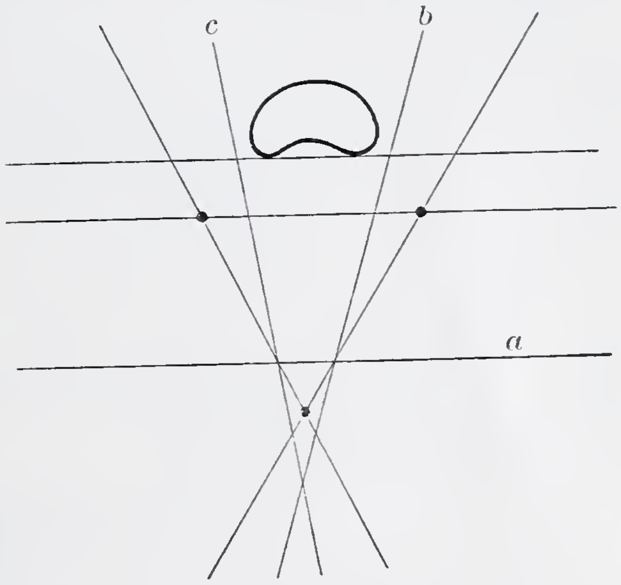
---

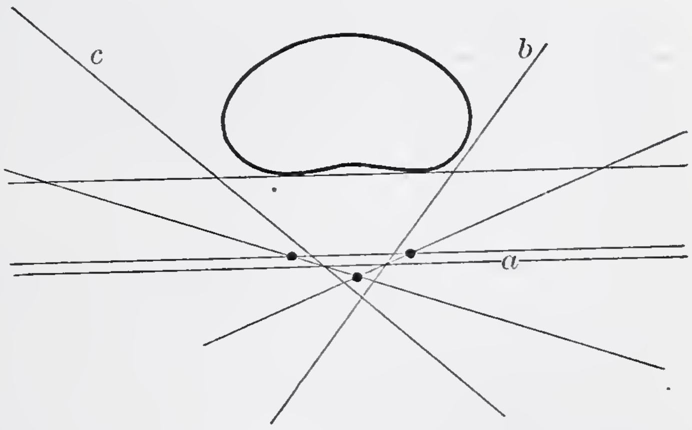
* $(y - x^2)^2 = \frac{5}{6}y^3 + (A - \frac{5}{6})x^2y^2 - \dfrac{y^4}{6}$ when put into the form of equation (9).

and the real circuit:  . If, on the other hand, $g^2 + 4lm^2 < 0$, the radicand is negative for every positive y and for every negative one whose modulus is greater than mod $\left(\frac{-4m(md+g)}{g^2+4lm^2}\right)$; hence the real circuit must lie below the axis $y = 0$, the dt lying still farther below: 

The dt thus lies between the circuit and the *concavity* of the oscnode. This, however, is a mere matter of analysis, since if the dps constituting the oscnode are really consecutive, the eye cannot distinguish the direction of curvature.

These curves with an isolated oscnode may be regarded as limits of those curves of type 18 which have not more than one non-isolated dt, when the three acnodes become consecutive; for instance, Fig. 18_c projects into the form here shown, in which a, b, c are the three isolated dts, a being the one that was entirely outside of the fundamental triangle in Fig. 18_c. Now let the three acnodes approach, as in the next figure. In the limit these three acnodes, consecutive, form the isolated oscnode and are no longer to be ranked as acnodes (see Art. **48**); the three isolated dts become consecutive dts with real contacts. Placing these curves in the table (Art. **2**) under (1), (2), II, not under (12), (13), we find that oscnodal quartics fully agree with the table.





B. *Quartics with a tacnode-cusp.*

xxv $\equiv$ B.

56. If in equation (9) we have

$$md + g = 0,$$

the equation may be written

$$(y - mx^2 - cxy - d_1 y^2)^2 = y^3(Ax + By), \quad . \quad . \quad (10)$$

where $d_1 \equiv \frac{d}{2}$, $A \equiv h + cd$, $B \equiv \frac{d^2 + 4l}{4}$. Hitherto l and h have been unrestricted, hence A and B are now unrestricted except that $A \neq 0$ to prevent degeneration.

The expansions at xy are

$$y = mx^2 + cmx^3 \pm m\sqrt{mA}x^{\frac{7}{2}},$$

and the point is a tacnode-cusp (Art. **3**). The expansions are real on one or the other side of $x = 0$ whether Am be positive or negative.

Type 36 ≡ xxv ≡ B. Fig. 36_a represents the whole type: an extra bay is impossible, the two dts being at the tacnode-cusp, which also involves one of the real inflexions. $I = 2$, hence (15), II of the table (Art. **2**) is not represented among quartics with a tacnode-cusp.

The type is the limit of type 33 when a loop becomes evanescent, or the limit of type 7 when both crunodes move up to the cusp. It may of course touch or cut $x = 0$; for instance, as in Fig. 36_b, roughly the locus of

$$(y - x^2 - 3y^2)^2 = y^3(x + 4y).$$

If the transition be made without recourse to symmetry, type 36 is the transition curve from a quartic with a non-isolated oscnode to one with an isolated oscnode; for instance, Fig. 33_b, $md + g > 0$; Fig. 36_b, $md + g = 0$; Fig. 35_a, $md + g < 0$.

§ IV. Quartics with three coincident double points; that is, with a triple point.

57. The triple point being at xy, the most general equation of the quartic is

$$ax^4 + by^4 + fx^3y + gx^2y^2 + hxy^3 + kx^2y + dy^3 + lx^3 + mxy^2 = 0.$$

One of the tangents at xy *must* be real; let it be

$$y = 0.$$

Choose $x = 0$ so that the other two tangents may be

$$dy^2 + kx^2 = 0.$$

The equation becomes

$$ax^4 + by^4 + fx^3y + gx^2y^2 + hxy^3 + kx^2y + dy^3 = 0, \quad . \quad (11)$$

in which $a \neq 0$ and either $b \neq 0$ or $d \neq 0$ to prevent degeneration.

xxvi. *Quartics with a triple point of the first kind.*

d and k have opposite signs.

58. Modifying the definition of the descriptive terms 'divided' and 'non-divided' (Art. **12**) to suit the present case, any one of the three pairs that can be formed of the three tangents at the triple point will be called 'non-divided' when an even number of branches of the curve pass between them; 'divided,' when an odd number of branches pass between them.

Type 37. Quartic with only non-divided tangents at the triple point of the first kind. This is the limit of type 1 or type 3 when the lines joining the crunodes become *concurrent.*

A simple example is

$$x^4 + y^4 + gx^2y^2 + 2x^2y - 2y^3 = 0,$$

which for $-2 < g < -1$ has a bay on one loop (Fig. 37_b), and for $g > -1$ has an isolated dt (Fig. 37_c).

To obtain the relations among coefficients of equation (11) that distinguish this type, we proceed as follows: considering figures 37 it is evident that a and k must have the same sign, the shape of the branch tangent to $y = 0$ being given by $ax^2 + ky = 0$; the tangents to the other two branches being $x = \pm \sqrt{\frac{-d}{k}}\, y$, for y small and positive equation (11) must have one real root between $x = 0$ and $x = +\sqrt{\frac{-d}{k}}\, y$, and one between $x = 0$ and $x = -\sqrt{\frac{-d}{k}}\, y$. These conditions demand respectively that

$$D_1 \equiv b + a\frac{d^2}{k^2} - g\frac{d}{k} + \left(h - f\frac{d}{k}\right)\sqrt{\frac{-d}{k}}$$

and $$D_2 \equiv b + a\frac{d^2}{k^2} - g\frac{d}{k} - \left(h - f\frac{d}{k}\right)\sqrt{\frac{-d}{k}}$$

have the opposite sign to d. If the curve had been turned

 the conditions would have been that a and k should have opposite signs and D_1, D_2 should have the same sign as d. Now, d and k having opposite signs (since the type comes under xxvi), the distinguishing feature for type 37 is seen to be that

$$a, D_1, D_2$$

have the same sign.

Type 38. Quartic with two pairs of divided tangents at the triple point of the first kind. One of the constants

$$a, D_1, D_2 \quad \text{(see type 37)}$$

differs in sign from the other two.

The locus is a double-odd circuit, the limit of type 4 or type 5 when the lines joining the dps become *concurrent*. All four dts are imaginary, as they must be for every quartic with a double-odd circuit except one with a tacnode.

The example given for type 37 serves also for the present type. If $g = -2$ the curve degenerates, and for $g < -2$ the locus is of type 38. The loop is, however, cut by infinity. Changing the sign of y^4, to bring the loop entirely into the finite, and modifying the numerical value of g so as to have the proper relation of signs for a, D_1, D_2, we have

$$x^4 - y^4 + gx^2y^2 + 2x^2y - 2y^3 = 0, \qquad g < 0,$$

the locus of which is a typical figure (Fig. 38). $I = 2$.

xxvii. *Quartics with a triple point of the second kind.*

$$d = 0.$$

59. The shapes at xy are given by

$$ky + ax^2 = 0,$$
$$by^3 + kx^2 = 0,$$

which lie on the same or on opposite sides of $y = 0$, according as a and b have the same or opposite signs.

Type 39. Quartic whose cuspidal tangent and ordinary tangent at a triple point of the second kind are non-divided.

The coefficients a and b have the same sign.

This is the limit of type 6 when the lines joining the dps become concurrent, or the limit of type 37 when a loop becomes evanescent.

The second dt may be non-isolated [e.g., Fig. 39_a, locus of $(y + \frac{3}{4}x)^2(y + \frac{4}{3}x)^2 = \frac{25}{6}x^2y(x - 1)$], or isolated [e.g., Fig. 39_b, locus of $x^4 + y^4 + x^2y = 0$].

Type 40. Quartic whose cuspidal tangent and ordinary tangent at a triple point of the second kind are divided.

The coefficients a and b have opposite signs.

The locus is a double-odd circuit, the limit of type 8 when the lines joining the dps become concurrent, or of type 38 when the loop becomes evanescent.

A simple example is

$$x^4 - y^4 + x^2y = 0. \quad \text{(Fig. 40.)}$$

This curve is the last of the double-odd circuits among unicursal quartics. Here, as in preceding cases, each of the two real inflexions has full range of its half-circuit. It is perhaps worthy of notice that, although the figures shown in Plate II present no turnings-back in the neighborhood of the real inflexions, such are possible. The present type, for instance, might have the form of the adjoined figure, which, though not drawn to scale, has the general appearance of the curve

$$x^4 - 14y^4 + 13x^2y^2 + 30x^2y = 0.$$

xxviii. *Quartics with a triple point of the third kind.*

$k = 0.$

60. *Type 41* $\equiv$ xxviii. The shape at xy is $ax^4 + dy^3 = 0$, recognized in the analysis of higher singularities as one crunode and two cusps, the triple point not differing visibly from an ordinary point except by a certain sharpness of curvature. The circuit can be brought entirely into the finite, and, since it has but one dt, cannot have more than one bay. The curve is therefore to the eye simply an 'oval,' embayed or not.

The simplest equation of the type is

$$y^3 = x^4,$$

which has an undulation at infinity, or in trilinears at xz.

$$(x^2 - y^2)^2 - dy^3z = 0$$

is a simple illustration of the embayed curve (Fig. 41_a), $z = 0$ being the dt.

$$(x^2 + y^2)^2 - dy^3 = 0$$

has infinity as an isolated dt, contacts being at the circular points (Fig. 41_b).

The type is the limit of type 9 when the lines joining the dps become concurrent.

xxix. *Quartics with a triple point of the fourth kind.*

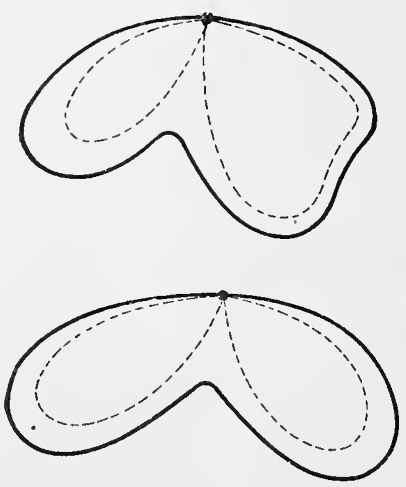
d and *k* have the same sign.

61. *Type 42* ≡ xxix. Since the triple point involves an acnode, the curve may perhaps best be considered (as other acnodal forms) as derived from a cuspidal form, the types of xxvii serving for this purpose.

If we place

$$ax^4 + by^4 + gx^2y^2 + fx^3y + hxy^3 + kx^2y \equiv F,$$

the types of xxvii are represented by $F = 0$, those of xxix by $F + dy^3 = 0$, d having the sign of k, which may be taken positive.



There is no difficulty in seeing that if a and b have the same sign, the locus $F + dy^3 = 0$ for d small lies just outside a figure of type 39, touching it at xy, where the cuspidal point is replaced by the acnode. The two dts (real) of the cuspidal figure pass into two real dts of the acnodal figure, one at least being non-isolated at first, and both being so if the figure of type 39 from which it is derived has an embayed loop. The two *new* dts are evidently *imaginary:* this is the first unicur-

sal quartic found thus far that has both imaginary and real, distinct dts (types 23, 24, 36 have two imaginary and two real, *consecutive* dts).

Similarly, if a and b have opposite signs the locus of $F + dy^3 = 0$ lies just *inside* a figure of type 40. Here, at last, the double-odd circuit disappears and this curve projects into the adjoined. The two new dts are real and non-isolated at first; the two that were imaginary in type 40 remain imaginary. With changing values of constants one or both real dts may become isolated.

Bearing in mind that the sign of b is important in type-characterization only when $d = 0$, in which case the curve must degenerate in order that b may change sign, it is clear that the derivatives from types 39 and 40 constitute but one type: this type has two imaginary and two real dts; the latter may be both non-isolated (Fig. 42_a), one isolated (Fig. 42_b), or both isolated (Fig. 42_c). (These three figures are drawn for a differing in sign from d and k.)

Quartics with a triple point exist for all the cases provided for by the table (Art. 2).

§ V. Quartics with one real and two imaginary double points.

62. Passing on to consider quartics having two imaginary dps, these may be taken at the circular points at infinity, since they can be made so by a real projection: the equation may then be written

$$(x^2+y^2)^2+(x^2+y^2)(ax+by)+(ex^2+2fxy+gy^2)+(hx+ly)+m = 0.$$

If a third dp be at xy the equation becomes

$$(x^2 + y^2)^2 + (x^2 + y^2)(ax + by) + (ex^2 + 2fxy + gy^2) = 0.$$

Since any pair of rectangular axes form a harmonic set with $x^2 + y^2 = 0$, we may choose $x = 0$ so that the tangents at xy are

$$gy^2 + ex^2 = 0,$$

whence the equation becomes

$$(x^2 + y^2)^2 + (x^2 + y^2)(ax + by) + ex^2 + gy^2 = 0, \quad . \quad (12)$$

in which $e \neq g$ to prevent degeneration.

The imaginary dps are cusps if

$$\alpha \equiv (a^2 - 4e) - (b^2 - 4g) = 0 \quad \text{and} \quad \beta = ab = 0.$$

The tangents from xy,

$$(b^2 - 4g)y^2 + 2abxy + (a^2 - 4e)x^2 = 0,$$

are real if $\gamma \equiv a^2g + b^2e - 4eg > 0$.

A. *Bicircular* quartics with a real double point.*

Either α or β or both must be different from zero.

xxx. *Bicircular quartics with a crunode.*

e and g have opposite signs.

63. Zeuthen, in the paper already referred to, shows that two imaginary nodes absorb two dts of the first species, thus leaving two of the first species; hence, since a unicursal quartic cannot have dts of the second species, the remaining two of the four dts existing for xxx must be imaginary.

Type 43. Crunodal bicircular quartic with real tangents from the crunode. $\gamma > 0$.

The figures are essentially those of type 27 without the isolated tacnode, but the curve may cut both axes going through the crunode, whereas the isolated tacnode in type 47 prevents one axis from being cut. $I = 2$. The type may be represented by the Lemniscate of Bernouilli, $(y^2 + x^2)^2 = 2c^2(x^2 - y^2)$.

Type 44. Crunodal bicircular quartic with imaginary tangents from the crunode. $\gamma < 0$. There are two joined loops, one being internal, so that one dt is non-isolated; the other is either non-isolated (Fig. 44_a) or isolated (Fig. 44_b).

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* *Bicircular* is here confined to quartics with *nodes* at the circular points.

xxxi. *Bicircular quartics with a cusp.*

$$e = 0.$$

64. *Type 45.* Cuspidal bicircular quartic with real tangents from the cusp. $\gamma > 0$. Limit of type 43 when a loop becomes evanescent. The two dts are imaginary, the real ones of type 43 having been absorbed by the tangents from the cusp. The locus is an *out-cusped* oval similar to Fig. 28 without the isolated tacnode. $I = 2$.

Type 46. Cuspidal bicircular quartic with imaginary tangents from the cusp. $\gamma < 0$. Limit of type 44 when the inner loop becomes evanescent. The *imaginary* dts of type 44 are absorbed. The locus is an *in-cusped* oval which may have a bay (Fig. 46_a) or not (Fig. 46_b).

xxxii. *Bicircular quartics with an acnode.*

e and g have the same sign. Two dts are imaginary, as in xxx.

65. *Type 47.* Acnodal bicircular quartic with real tangents from the acnode. $\gamma > 0$. Derivative from type 45 when the cusp is replaced by an acnode. The two real dts set free may be both non-isolated (Fig. 47_a), or one or both may be isolated (Figs. 47_b, 47_c, respectively); these are similar to figures of type 29 without the isolated tacnode.

Type 48. Acnodal bicircular quartic with imaginary tangents from the acnode. $\gamma < 0$. Derivative from type 46 when the cusp is replaced by an acnode. The two real dts may be both non-isolated (Fig. 48_a), one isolated (Fig. 48_b) or both isolated (Fig. 48_c).

B. *Cartesians* with a real double point.*

$$\alpha = \beta = 0.$$

66. The tangents from xy become $y^2 + x^2 = 0$; that is, they are absorbed by the lines joining the real dp to the imaginary cusps. The one dt is necessarily real.

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* As suggested by Salmon (*Higher Plane Curves*, Art. 280), the term *Cartesian* is here used to designate any quartic with *cusps* at the circular points at infinity, although Descartes' method of generating the curve is not always applicable.

xxxiii. *Crunodal Cartesian.*

e and g have opposite signs.

67. *Type 49* $\equiv$ xxxiii. The locus may in a way be considered as the limit of type 44 when the imaginary nodes become imaginary cusps, but the *bay* of Fig. 44_a is not possible when the nodes become cusps, since the curve has but one dt, which is necessarily non-isolated and bridges the indentation occasioned by the inner loop. $I = 0$. The Limaçon de Pascal (Fig. 49) may be taken as typical.

xxxiv. *Cuspidal Cartesian.*

$e = 0$.

68. *Type 50* $\equiv$ xxxiv. The locus is an *in-cusped* oval without a bay, the limit of type 49 when the loop becomes evanescent. $I = 0$. The Cardioid may be taken as typical.

xxxv. *Acnodal Cartesian.*

e and g have the same sign.

69. *Type 51* $\equiv$ xxxv. The locus, derivative from type 50 when the cusp is replaced by an acnode, consists of an acnode inside an oval which may have one bay (Fig. 51_a) or none (Fig. 51_b).

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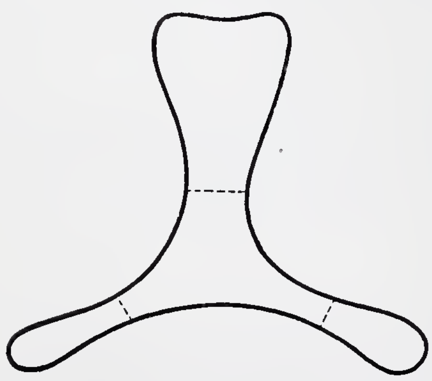
70. Having all possible types of unicursal quartics, it may not be amiss to examine their direct derivation from the non-singular curves by Zeuthen's processes (Art. **9**).

If Zeuthen's species 1 gradually change until the inner oval disappear (or, what is the same thing, if his species 2 change so that both ovals disappear), and three bays of the quadrifolium bend inward in such a way that each of the three eventually unites with the other two, the unicursal type 1 with the embayed loop (Fig. 1_a) results. Similar behavior on the part of a trifolium would give the same type without the bay on the loop.



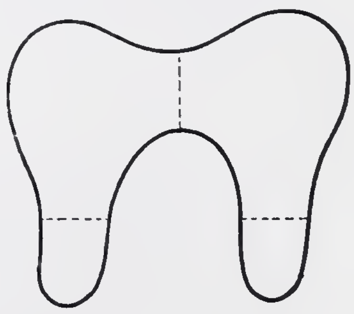
Again, if in the representative figure given for Zeuthen's species 9 (Plate I, Fig. 9) a line passing between the unifolium and the two ovals be sent to infinity, thus

bringing the convexity of the unifolium into proximity to the trifolium, and if ϕ be passed through the vertices of the triangle containing the trifolium, the constants being kept such that the ovals do not vanish, in the limit the ovals and the unifolium join to the trifolium, giving this same unicursal type 1; the same type without the bay would result from Zeuthen's species 19.

If in Figs. 3, 5, 7, I,* ϕ be passed through three vertices of the quadrilateral,† type 2, II will be obtained with three different positions of the real inflexions: one on each loop; one on a loop, one not; neither on a loop. If in addition to passing through the three vertices ϕ *touch* one of the two sides intersecting in the fourth vertex, the real inflexion belonging in the neighborhood of the vertex at which the contact occurs (the contact is necessarily at a vertex) will fall at the vertex.

The general method of obtaining a flecnode is to pass ϕ through the intersection of two dts and make it touch one of them. This would seem, also, to be the *only* method. According to Plücker (*Theorie der algebraischen Curven: Doppeltangenten der Curven vierter Ordnung.* Art. 102) if the conic be two real right lines and a dt pass through their intersection a flecnode results, but when the dts are of the first species this seems rather to give an *undulation*, not a dp at all. If ϕ be two right lines L_1, L_2 and one of them pass through *tu* a dp results at *tu*, but no inflexion falls there unless *both* lines L_1, L_2 pass through *tu*; in this case *tu* is a *bi*-flecnode, *t* and *u* being the inflexional tangents. Moreover, ϕ *must* be degenerate in order to obtain a biflecnode, since a proper conic cannot pass through the intersections of two lines and touch both of them. If, therefore, $tuvw = k\phi^2$ be the equation of a quartic with a biflecnode, according as ϕ consists of two real or two imaginary right lines, it will cut all dts of the quartic in real points or all in imaginary points. This origin of a biflecnode accounts for certain limitations in the quartic when it possesses such a dp. [See Arts. **28** (type 13), **31**, **74**.]

Type 2 may also be regarded as resulting from a quadrifolium when one bay bends deeply inward and joins to each of the other three bays.



If three bays of Fig. 1, I, bend inward and meet the inner oval, there results type 3, II with $I = 2$. The same type results from the passage of ϕ through the vertices of the tri-

* The large Roman numerals refer to the number of the plate.

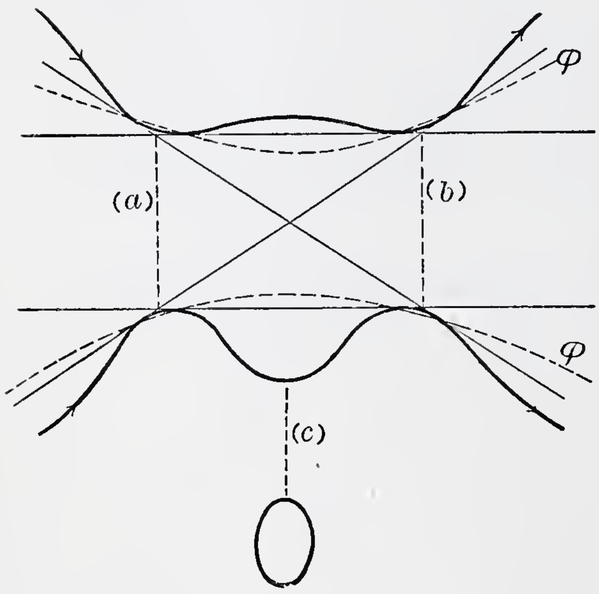
† 'The quadrilateral' or 'the triangle,' without further specification, will designate the quadrilateral or the triangle whose sides are cut by ϕ in the figures of Plate I.

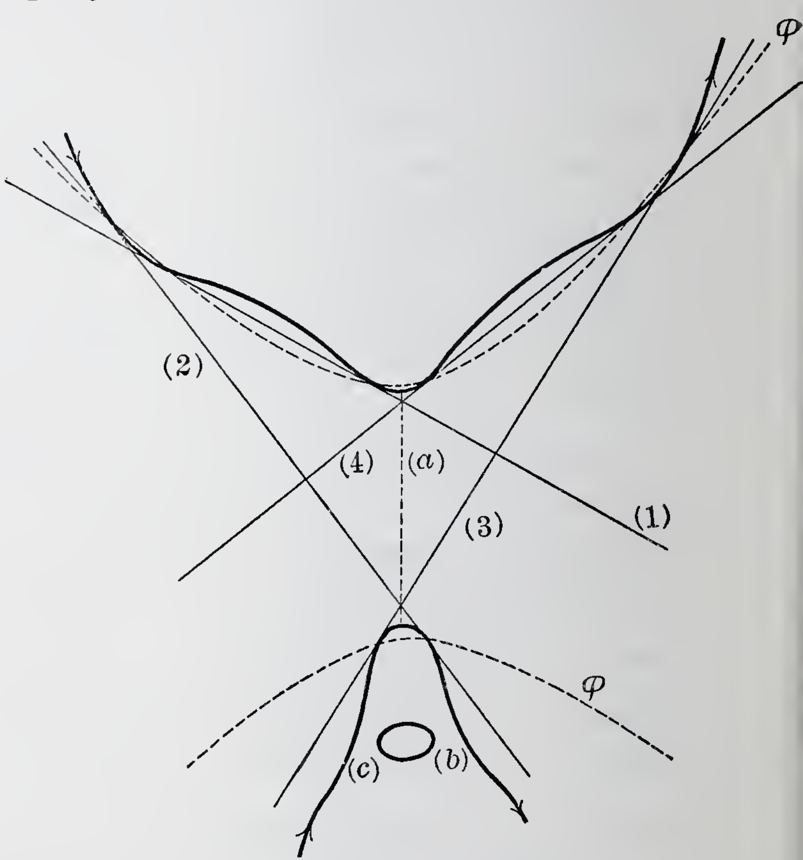
angle in Fig. 8, I. Similarly Figs. 10, 18, I, would give type 3, II without the bay [e.g., Fig. 3_c, II].

If Fig. 2, I, be projected so that but one oval is cut by infinity and ϕ be passed through the three vertices which lie opposite the non-divided oval and the two portions of the divided one, Fig. 4_b, II is obtained. The same type with the real inflexions differently situated can be obtained from Figs. 4, 6, I, by passing ϕ through three vertices (the convexity of the unifolium having been first brought by projection into proximity to the trifolium of Fig. 4, I).

Type 5, II does not seem to be obtainable by the mere passing of ϕ through vertices. If in Fig. 6, I, two dts of the first species which cut out of ϕ a portion from which the quartic is excluded (in other words, two which are dts to a *bifolium* in Fig. 7, I) be moved up to coincidence, the oval being swept out of existence, and if at the same time ϕ be passed through the intersection of the other two dts of the first species, Fig. 5_c, II is obtained. The same type with different positions of the inflexions is similarly obtainable from Figs. 4, 9, II.

Types 4, 5, II can also be obtained in other ways. For instance, suppose species 1, I, be projected so as to bring all four dts





of the first species into a narrow portion of the plane between two portions of the quadrifolium, as in the adjoining figures, and suppose the spaces (a), (b), (c) be made to close up by the joining of

the portions of the curve whose union is thus indicated. Types 4, 5, II, respectively, would be obtained. Consider the effect on the dts and on ϕ when the space (a) is closed up in the right-hand figure. It is evident that dts 1, 2 and 3, 4 come into coincidence, forming two abnodal tangents. Zeuthen describes the production of such a dp as resulting from making the dts coincide in pairs and *passing ϕ through their point of intersection.* It seems, however, that if

$$tuvw = k\phi^2$$

represent the non-singular curve, t, u, v, w being the dts and ϕ the conic through their contacts, the coincidence of the dts in pairs, giving

$$t^2v^2 = k\phi^2,$$

must cause degeneration into two conics

$$tv = \pm \sqrt{k}\phi$$

unless ϕ become identical with the dts, making the equation illusory. Careful examination of the figure shows that ϕ does degenerate into two right lines which become respectively coincident with the two abnodal tangents arising from the two pairs of coinciding dts, for dts 1, 2, for instance, cannot coincide without causing that portion of ϕ which lies between them to become a right line coinciding with them. In the figure for producing type 4, II, when (a), (b), are closed all four dts of the first species coincide and ϕ becomes two coincident right lines which also coincide with the four original dts of the first species. If, therefore, the equation in the form

$$tuvw = k\phi^2$$

be desired for types 4, 5, II, it is to be obtained from the former methods given for derivation of these types.

To obtain any cuspidal form directly from the non-singular curve, it is necessary so to arrange the constants that the oval which for the corresponding crunodal form joins to another circuit and gives a *loop,* shall gradually diminish and arrive at the vertex as an *acnode* just as ϕ reaches the limit and passes through the vertex: the acnode, joining to the non-singular circuit, gives the cusp.*

* Plücker (*Theorie der algebraischen Curven,* under *Doppeltangenten der*

To obtain an acnodal form, omit *that step* in the preceding processes which caused the union of an oval with another circuit, and let the oval shrink to a point.

71. For higher singularities than the ordinary dp the processes may appear somewhat complex at first; if, however, the process described for obtaining any one of the forms with three real, distinct dps be combined with that previously described for obtaining from it a form endowed with a higher singularity, the combination will give a quartic with the higher singularity. A few illustrations may be given.—Type 19, II comes, among other ways, from Fig. 3, I when three of the dts of the first species are concurrent and ϕ is tangent to one of them at the point of concurrence and passes through the intersection of the fourth with one of those not touched by ϕ. (The line joining the *embrassement* to the crunode of type 19, II replaces one of the dts of the first species in Fig. 3, I, the tangent from the *embrassement* to the loop connected with it replaces another, and the tangent *at* the *embrassement* replaces the third of the three that are made concurrent, this being the one touched by ϕ.) Again, if in the derivation of type 5, II from Figs. 4, 6, 9, I ϕ be so varied during the approach of the two dts as to *touch* them when they reach the limit, type 24, II may be obtained. If in Fig. 9, I, the three dts which enclose the trifolium be made concurrent by moving one up to the intersection of the other two, keeping ϕ in position, in the limit ϕ becomes two right lines passing through the point of concurrence, and the quartic is of type 37, II, with or without the bay, according as ϕ degenerates into two real or two imaginary lines. Again, suppose species 1, I, be projected somewhat as for the derivation of type 5, II (Art. **70**), the dts 2, 3 being isolated instead of cutting ϕ. The convexities to which the oval joins to give type 5 are thus no longer present; but if, while the opening (a) gradually closes up (all four dts approaching coincidence), the oval move up toward the outer circuit and

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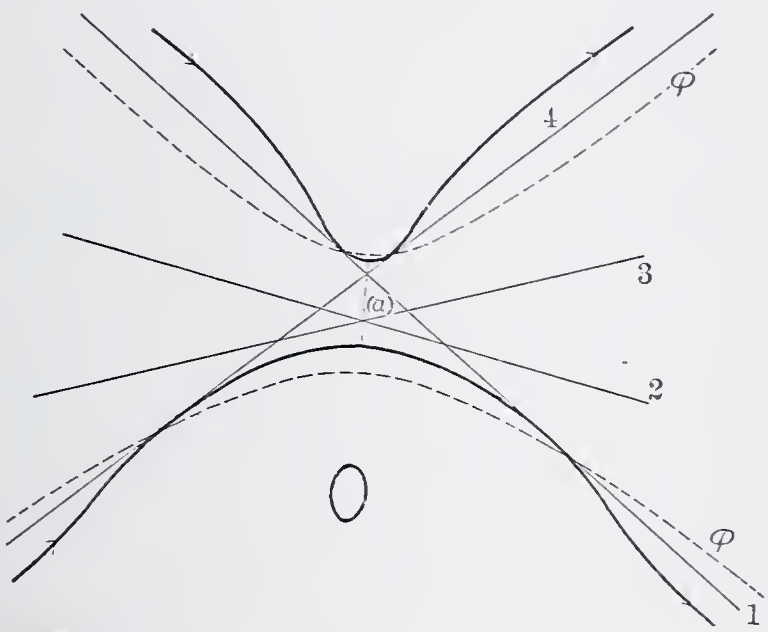
Curven vierter Ordnung) gives an ordinary cusp as arising from the concurrence of three of the four dts which appear in his general equation

$$pqrs + \mu\Omega_2^2 = 0$$

of the quartic, but these dts are not always of the ***first species***, so that his method and the one here given are not contradictory. The same remark is applicable to some of the other derivations here given for singular quartics.

arrive at its limit with the dts, type 38, II is obtained; if the oval become a point just as the varying quantities reach their limits, type 40, II results. The dts unite to form one of the ordinary tangents at the triple point. If the isolated dts and the oval reach the limiting position before the arrival of the rest of the curve, an ordinary *embrassement* is obtained unless the oval shrink into a point at the same instant, in which case type 42, II results.



72. The similarity between certain quartics with two imaginary nodes and those with an isolated tacnode has been noted; this is accounted for by the fact that they come from the same non-singular quartics by a very similar process:—for instance, suppose in Fig. 25, I, ϕ be passed through the intersection of the two non-isolated dts and the two isolated dts be made to move up together; if the two ovals disappear before the dts reach the limiting position, the dts become the join of two imaginary nodes formed by the coinciding contacts of the two isolated dts (each contact on one dt coinciding with one on the other dt), giving type 43, II; if, however, the ovals do not disappear, but only shrink each to a point by the time the dts reach the limit, the latter become the consecutive dts at an isolated tacnode, giving type 27, II. Types 44, 46, 48, II, having no counterpart among quartics with an isolated tacnode, arise from annular quartics having two isolated dts which coincide and give imaginary nodes. If there be three isolated dts and they be made to become consecutive, three ovals being reduced to points, an isolated oscnode is obtained; in this way Figs. 32, 36, I, for in-

stance, give an isolated oscnode. If the ovals disappear or if the quartic be annular, the three isolated dts become in the limit the join of two imaginary cusps; for instance, Fig. 29, I, gives type 49, II if the inner oval join to the bay of the outer circuit and the three isolated dts coincide.

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73. Had the attempt been made to derive all singular quartics directly from the Zeuthen figures, the processes, conditions regarding behavior of ovals, etc., necessary to produce the types would often have been sufficiently complicated to demand some elementary explanation showing the compatibility of the conditions. Hence it was thought that an analytical investigation of unicursal quartics, letting the same general method of investigation and the same general principles of type distinction be applied throughout, would be a more satisfactory method.* For the remaining two grand divisions of singular quartics—those presenting respectively one and two dps—the assurance that the conditions necessary to give the different types presented can be simultaneously imposed is derived from the fact that, for each type, there is at least one *unicursal* quartic whose derivation from the non-singular quartic demands the same conditions and *others in addition.* Hence analysis will be omitted for the remaining forms.

74. For curves admitting more than one real circuit, the nature and position of dps and tangents from them do not constitute sufficient basis for classification, since the bays may be so differently distributed to the circuits and even to the parts of the same circuit. On the other hand it is desirable to avoid a great multiplication of types and figures. To this end the following plan is adopted:—Only the *maximum number of bays and the maximum number of real circuits* will be given for any combination of dps; *the dis-*

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* I do not overlook the analytical treatment of unicursal quartics by means of their parametric expression, given by Brill (*Math. Ann.*, XII, 1877). The plates to Brill's paper illustrate the principal types of *nodal* quartics (omitting, however, one distinct oscnodal type,—type 34 of those here enumerated), but not in general the cuspidal forms; the bays are indicated only in very few cases, the process adopted not lending itself conveniently to their determination; the author does not profess to indicate all possible inflexions, but it is perhaps well to notice that his curves 6, 7, 8, 6^a, 8^a depend for their existence on certain real inflexions confined within certain limits.

tribution of bays will be indicated; for any given type only one position will be shown for a real inflexion which, not being coupled with another in a bay, cannot pass off, but may pass through a crunode situated in its neighborhood. The following summary of forms presenting one and two dps is therefore to be understood as follows: Every type presented may be varied by the removal of any bay or bays, or any non-singular circuit or circuits (with the exception of a circuit which, though non-singular *per se*, helps to form a dp by crossing or touching another circuit), or by both these processes; a real inflexion that is described as not being in a bay and as being capable of lying on either side of a crunode may have the special position *at* the crunode: if, however, the curve be so specialized as to have a *bi*-flecnode other limitations arise. (See Art. **70**.)

The only quartics admitting a biflecnode at all are certain double-odd circuits (types 5, 13, 24, II and 7, IV), the quartics consisting of two odd circuits (types 7, III and 8, 13, IV), and certain 'double-loop' circuits. A double-loop circuit endowed with a biflecnode must have one bay on each loop or else no bay at all; if the quartic have also a non-singular circuit this must have two bays or none; if one of the bays in either case be an undulation the other also must be an undulation. These and other properties concerning the relative position of contacts of dts, etc., are readily deducible from the fact that ϕ must be two right lines (real and distinct, imaginary, or coincident) in order to give rise to a biflecnode (Art. **70**).—The only quartics admitting *two* biflecnodes are types 13, II and 4, 7, IV.

III. QUARTICS WITH ONE DOUBLE POINT.*

i. *Quartics with one crunode.*

75. *Type 1.* A single circuit with a loop turned inward, three bays on outer portion. (Fig. 1, I†, inner oval joined to a bay of quadrifolium.)

Type 2. One double-loop circuit *with two bays on the same loop*, and one oval. The two real inflexions not in a bay may lie both on the embayed loop or one on each loop, but apparently not *both* on the *non-embayed* loop, so long as the other loop remains embayed. (Fig. 2, I, ϕ through vertex between oval and quadrifolium; Fig.

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* See Plate III.

† The Zeuthen figures from which the types come when all bays and circuits are present will be given. Corresponding ones for fewer bays and circuits can readily be selected.

4, I, ϕ through vertex between trifolium and embayed side of unifolium.)

Type 3. One double-loop circuit without a bay, and two unifolia. The two inflexions on the singular circuit may be one on each loop or both on one loop. (Fig. 3, I, ϕ through any vertex; Fig. 5, I, ϕ through vertex between bifolium and oval.)

Type 4. One double-loop circuit without a bay, one bifolium, and one oval. The two inflexions on the singular circuit may be one on each loop or both on one loop. (Fig. 5, I, ϕ through vertex between the two unifolia; Fig. 7, 1, ϕ through a vertex between bifolium and oval.)

Type 5. One double-loop circuit with a bay on one loop, one unifolium, and one oval. The two real inflexions not in a bay may be one on each loop or both on the embayed loop, but apparently not both on the *non-embayed* loop, so long as the other loop remains embayed. (Fig. 5, I, ϕ through vertex between bifolium and unifolium; Fig. 9, I, ϕ through a vertex between trifolium and an oval.)

Type 6. One double-loop circuit with a bay on each loop, and two ovals. The two real inflexions not in a bay may be one on each loop or both on one loop. (Figs. 7, 9, I, ϕ through the vertex between the embayed circuits.)

Type 7. Two odd circuits that cross in one point, and one oval. This is the only type of quartic with but one dp which *must* cut infinity; it corresponds to the double-odd circuit appearing in unicursal quartics, in that it must cut infinity twice and is derived from the non-singular quartic by making the dts of a quadrifolium coincide in pairs (after the manner of the derivation given (Art. **70**) for unicursal type 5, except that the oval is not joined to the other circuits); but it is *not* a double-odd circuit, but simply two odd circuits: three real inflexions must lie on each of the odd circuits, but they have a wide range of position along the circuit; all dts are imaginary, all six abnodal tangents are real when the oval is present, otherwise four are imaginary.

The equation $tuvw = k\phi^2$ becomes illusory if t, u, v, w were the dts of the first species in the non-singular quartic (see Art. **70**), but since a quartic can be put into this form in more than one way the equation of the type may still be found in this *form*, with other lines as t, u, v, w.

ii. *Quartics with one cusp.*

76. *Type 8.* A single circuit, *in-cusped,* with three bays. Limit of type 1, III.

Type 9. One *out-cusped,* doubly embayed circuit, and one oval. Limit of type 2, III.

Type 10. One *out-cusped* circuit without a bay, and two unifolia. Limit of type 3, III.

Type 11. One *out-cusped,* non-embayed circuit, one bifolium, and one oval. Limit of type 4, III.

Type 12. One *out-cusped* circuit with one bay, one unifolium, and one oval. Limit of type 5, III.

It is to be noted in the preceding that when three bays are on one circuit there is but that one circuit: when two bays are on the singular circuit and are not separated by the dp only one other circuit—an oval—is possible: otherwise there may be three real circuits in all.

iii. *Quartics with one acnode.*

77. All types of quartics with one acnode as the only singularity can be seen from Plate I by simply imagining an oval to shrink to a point. Hence separate figures for these will not be given. Applying the convention explained (Art. **74**) for lessening the number of types, the types of acnodal quartics are :

Type 13. One quadrifolium and an acnode inside. (Fig. 1, I.)

Type 14. One quadrifolium, one oval, and an acnode. (Fig. 2′, I.)

Type 15. One bifolium, two unifolia, and an acnode. (Fig. 5, I.)

Type 16. Two bifolia, one oval, and an acnode. (Fig. 7, I.)

Type 17. One trifolium, one unifolium, one oval, and an acnode. (Fig. 9, I.)

Representative figures for these types 13–17, III can also be easily imagined from the figures for cuspidal curves, the cuspidal point being cut off for the acnode, as acnodal figures hitherto have been obtained. The figure thus obtained from Fig. 11, III, for instance, would be a projection of the one obtained from Fig. 7, I.

IV. QUARTICS WITH TWO DOUBLE POINTS.*

§ I. QUARTICS WITH TWO REAL, DISTINCT DOUBLE POINTS.

i. *Quartics with two real, distinct crunodes.*

78. *Type 1.* One bicrunodal circuit with a bay *which is not on a loop.* The two real inflexions not in the bay may be (α) one near each crunode but not on the loop; (β) one near one loop but not on it and one on the other loop; or (γ) one on each loop. (Fig. 2′, I, ϕ through the two vertices just opposite the ovals; Fig. 4, I, ϕ through vertex between trifolium and oval and through vertex between trifolium and embayed side of unifolium; Fig. 8, I, ϕ through the two vertices between bifolium and the unifolia.)

Type 2. One bicrunodal circuit without a bay, and one unifolium. The two real inflexions of the singular circuit may assume any one of the three positions described in the preceding type. (Fig. 3, I, ϕ through vertex between bifolium and oval and through the adjacent vertex between bifolium and unifolium; Fig. 9, I, ϕ through vertex between trifolium and the ovals.)

Type 3. One bicrunodal circuit with a bay on one loop, and one oval. The two real inflexions not in the bay may be (α) one on each loop; (β) neither on a loop; or (γ) one on the embayed loop, the other near the other loop but not on it; but apparently not 'one on the *non-embayed* loop, the other not on a loop,' so long as the bay remains. (Fig. 5, I, ϕ through vertex between bifolium and unifolium and through adjacent vertex between the unifolia; Fig. 9, I, ϕ through vertex between trifolium and unifolium and through another vertex; Fig. 7, I, ϕ through vertex between a bifolium and oval and through adjacent vertex between the bifolia.)

Type 4. Two double-loop circuits, non-embayed. The four real inflexions may be (α) one on each loop of each circuit; (β) one on each loop of one circuit, two on one loop of other circuit; or (γ) two on one loop of each circuit. (Figs. 3, 5, 7, I, ϕ through two opposite vertices.)

Type 5. Bifolium twice intersected by an oval. (Figs. 4, 9, I, two dts of first species made to coincide, the oval or ovals being swept out of existence by the approach of the dts.)

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* See Plate IV.

Type 6. One unifolium twice intersected by another. (Figs. 6, 7, I, two dts made to coincide, the oval or ovals disappearing.)

For types 5, 6 all abnodal tangents are imaginary, as in type 1, III, from which they are derivable by making the loop join to a bay.

Type 7. One bicrunodal double-odd circuit, and one oval. (Figs. 2, 4, 6, I, ϕ through two opposite vertices.) The dts are all imaginary. Two real tangents can be drawn from each crunode to the double-odd circuit and two to the oval. The real inflexions have range and limitations similar to those of type 13, II.

Type 8. Two odd circuits which intersect in one point, one of these odd circuits being crunodal *per se*. The derivation is the same as for type 7, III (Art. **75**), with the addition that the oval joins to one bay. The abnodal tangents are all four real from the intersection of the two circuits, all imaginary from the other crunode. The odd circuit which is crunodal *per se* has one real inflexion, the other circuit has three; these real inflexions may range from one side to the other of the crunode in which the two circuits intersect, etc.

ii. *Quartics with one crunode and one cusp.*

79. *Type 9.* A single circuit with a cusp, a loop, and a bay *not on the loop*. Limit of type 1, IV. One real inflexion may vary from one side of the crunode to the other.

Type 10. One circuit with a cusp and a loop but no bay, and one unifolium. Limit of type 2, IV. One real inflexion may vary from one side of the crunode to the other.

Type 11. One circuit with a cusp and an embayed loop, and one oval. Limit of type 3, IV.

Type 12. One double-loop circuit, and one non-embayed, *out-cusped* circuit. Limit of type 4, IV. The two real inflexions of the crunodal circuit may be one on each loop or both on one loop.

Type 13. Two odd circuits intersecting in one point, one of the circuits being cuspidal. Limit of type 8, IV. The cuspidal circuit has one real inflexion, the other circuit three; the real inflexions may range from one side of the crunode to the other, etc.

iii. *Quartics with two real, distinct cusps.*

80. *Type 14.* One embayed bicuspidal circuit. Limit of type 1, IV when both loops become evanescent.

Type 15. One non-embayed bicuspidal circuit, and one unifolium. Limit of type 2, IV when both loops become evanescent.

Type 16. Two non-embayed, *out-cusped* circuits. Limit of type 4, IV when one loop of each circuit becomes evanescent.

iv. *Quartics with one crunode and one acnode.*

81. Separate figures will not be shown for acnodal curves, since they can easily be imagined from preceding figures. Typical ones for the present division are obtainable from Figs. 2, 4, 5, 6, 7, III by the shrinking of an oval, these giving in order the following types:—

Type 17. One double-loop circuit with two bays on one loop, and an acnode.

Type 18. One non-embayed double-loop circuit, one bifolium, and an acnode.

Type 19. One double-loop circuit with a bay on one loop, one unifolium, and an acnode.

Type 20. One double-loop circuit with a bay on each loop, one oval, and an acnode.

Type 21. Two odd circuits intersecting in one point, and an acnode. The remarks concerning real inflexions, made for the types 2, 4, 5, 6, 7, III, apply respectively to types 17–21, IV.

v. *Quartics with one cusp and one acnode.*

82. Figures 9, 11, 12, III, by the shrinking of the oval, give as typical curves—

Type 22. One doubly embayed *out-cusped* circuit, and an acnode.

Type 23. One non-embayed *out-cusped* circuit, one bifolium, and an acnode.

Type 24. One *out-cusped* circuit with a single bay, one unifolium, and an acnode.

vi. *Quartics with two acnodes.*

83. All forms of quartics with two acnodes may be obtained from figures of Plate I by the shrinking of two ovals. As typical may be given—

Type 25. One quadrifolium and two acnodes. (Fig. 2′, I.)

Type 26. Two bifolia and two acnodes. (Fig. 7, I.)

Type 27. One trifolium, one unifolium, and two acnodes. (Fig. 9, I.)

§ II. Quartics with two consecutive double points.

A. *Tacnodal Quartics.*

vii. *Quartics with an embrassement.*

84. *Type 28.* One non-embayed circuit with an *embrassement,* and one unifolium. Limit of type 2, IV when the crunodes become consecutive.

Type 29. One circuit with an *embrassement* and one bay, and one oval. Limit of type 3, IV.

Type 30. One bifolium and an inner oval touching the bifolium. This is the limit of an annular quartic having two isolated dts, when the inner oval and the isolated dts all approach the same point on the outer circuit and arrive there together, the isolated dts becoming non-isolated and consecutive; or, it is the limit of type 5, IV when the oval moves so as finally to enter the bifolium and have contact with it.

viii. *Quartics with an opposition.*

85. *Type 31.* One bifolium and an outer oval which touches it. Limit of type 5, IV when the oval moves so as to be wholly outside the bifolium, having contact with it.

Type 32. Two bifolia in contact. Limit of type 6, IV.

ix. *Quartics with an isolated tacnode.*

86. The forms arise from those non-singular curves which have at least two isolated dts and two ovals lying between them in such position as to be forced up together as isolated points if they do not disappear before the limiting position is reached, when the isolated dts move up together (see Art. 72); that is, they arise from species 21, 25, 28, 31, 32, 35, 36, I. As typical may be taken—

Type 33. Two unifolia and an isolated tacnode. (From Fig. 25, I.)

Type 34. One bifolium, one oval, and an isolated tacnode. (From Fig. 28, I.)

B. *Quartics with a node-cusp.*

x ≡ *B*.

87. *Type 35.* One non-embayed node-cuspidal circuit and one unifolium. Limit of type 28, IV when one loop becomes evanescent.

Type 36. One node-cuspidal circuit with a single bay, and one oval. Limit of type 29, IV when the non-embayed loop becomes evanescent.

§ III. Quartics with two imaginary double points.*

xi. *Bicircular Quartics.*

88. *Type 37.* Two unifolia : similar to Fig. 33 without the isolated tacnode. (See Art. 72.) (From Fig. 25, I, when the two isolated dts coincide.)

Type 38. One bifolium and one oval : similar to Fig. 34 without the isolated tacnode. (See Art. 72.) (From Fig. 28, I, when the two isolated dts coincide.)

Type 39. One bifolium and one oval inside. (From Figs. 20, 23, 26, I, when the two isolated dts coincide.)

xii. *Cartesians.*

89. *Type 40* ≡ xii. One unifolium and one oval inside. (From Figs. 29, 30, I, when the three isolated dts coincide, the ovals disappearing from the non-singular quartic before the dts reach their limit. Figs. 31, 32, I, would give the same type without the oval. Figs. 33, 34, I, would give the Cartesian without the bay. Fig. 35, 36, I, would give the Cartesian without bay or inner oval. See Art. 72.)

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90. From the derivation of these singular quartics various groups of more than five points on the curve can be found through which a conic can be passed, thus affording a help to accuracy of outline of the quartic. For instance, a conic can be passed through the points specified in the following as affording a few simple examples: in types 2–6, 9–12, III the dp and the contacts of both

* Here, as before, the dps are taken at the circular points. Also 'bicircular quartic' and 'Cartesian' are used respectively when the imaginary dps are *nodes* and *cusps*.

'bay dts' and both abnodal tangents that touch the singular circuit; in types 1, 8, III the eight contacts of the real dts; in types 1–7, 9–12, IV both dps, the contacts of real abnodal tangents touching the circuit on which is situated the dp from which the tangents are drawn, and the contacts of 'bay dts'; in types 28, 29, IV four consecutive points on the curve at the *embrassement* and the contacts of the 'bay dt' and the two real tangents from the dp to the singular circuit; in types 30–32, IV four consecutive points on the curve at the tacnode and the contacts of the 'bay dts'; in types 35, 36, IV five consecutive points at the node-cusp and the contacts of the 'bay dt,' etc. In these illustrations the conic in question is the one that goes through the eight contacts of dts of the first species in the non-singular quartic. By selecting from the dts to the non-singular curve other groups of four through whose contacts a conic will pass, and observing what points in the singular curve correspond to these contacts, other groups of such points are found for the singular curve.

Under unicursal quartics it was mentioned that (9), (18), II of the table (Art. **2**) do not exist. It is also to be noted that the three divisions (7), (15), (19), IV of the table are not represented by any real curve.

91. It will be observed that the variations provided for, Art. **74**, although giving rise to a seemingly needless multiplication of types, do not give rise to so many as might at first appear, for the reason that several types, as given in the preceding work, may give rise to one and the same type through these variations. For instance, if from type 3, III a single bay pass off, and from type 5, III the bay pass off from the loop, the same description of curve is obtained; viz., one non-embayed double-loop circuit, one unifolium, and one oval. Similarly, several others in Plate III pass into each other; similarly in Plate IV. These types that may thus pass into each other have certain common features in the way of dts and abnodal tangents; for instance, to every double-loop circuit there are two dts of the first species which cannot become isolated, and two real abnodal tangents, etc. It is thought that the figures given in Plates III, IV will show these and other facts concerning the types with sufficient clearness to make further description unnecessary.

92. Although the object of this paper, as already stated, is to present the forms of quartics when projected so as to cut infinity the least possible number of times, the following summary of the relations to the curve that the line infinity may by projection of these figures be made to assume may not be amiss. (See also Plücker, *Theorie der algebraischen Curven, erster Abschnitt*, § 7.)

i. Infinity meets the quartic in four imaginary points, which may be—

[1] Four 1-fold points. Infinity cuts the quartic in four imaginary points.

[2] Two 2-fold points.

(*a*) Infinity is an isolated dt.

(*b*) Infinity joins two imaginary dps.

Note.—From division i types 4, 5, 8, 13, 23, 24, 25, 36, 38, 40, II; 7, III; and 7, 8, 13, IV are entirely excluded.

ii. Infinity meets the quartic in two imaginary and two real points; the real points may be—

[1] Two 1-fold points.—Infinity cuts the curve in two imaginary and two real points; the real intersections are on the same circuit or (in those types having two odd circuits) on different circuits.

[2] One 2-fold point.

(*a*) Infinity cuts the quartic in two imaginary points and is an ordinary tangent. (This is not possible in the curves enumerated in division i, note.)

(*b*) Infinity cuts the quartic in two imaginary points and passes through a real dp. The dp may be of any nature, but the combination (*b*) is not possible for all types of singular quartics: for instance, type 3, II does not admit it; other types will be apparent in looking over the plates.

iii. Infinity meets the quartic in four real points, which may be

[1] Four 1-fold points.—Infinity cuts the quartic in four real points, which may be all on one circuit; two on each of two circuits; two on one circuit, one on each of two others (possible in type 7, III); three on one circuit, one on another (in types 7, III and 8, 13, IV).

[2] Two 1-fold points and one 2-fold point.

(*a*) Infinity cuts the quartic twice and touches it at an ordinary point.—The contact and the intersections may be on one circuit; the contact on one circuit, the intersections on another; the con-

tact on one circuit, the intersections one on each of the other two circuits (possible in type 7, III); the contact and one intersection on one circuit, one intersection on another (possible in types 7, III and 8, 13, IV).

(*b*) Infinity cuts the quartic twice and passes through any real dp. The dp and the intersections may be on the same circuit; the dp on one circuit, the intersections on another; the dp and one intersection on one circuit, one intersection on the other circuit (in types 8, 13, IV); the dp common to two circuits, the intersections one on each of the two (in types 5, 6, 30–32, IV); the dp common to two circuits, the intersections on another (in type 7, III); the dp common to two circuits, the intersections on one of them (in types 7, III and 8, 13, IV).

[3] Two 2-fold points.

(*a*) Infinity is a dt, which may be of the first or second species, but if of the first species must of course be non-isolated.

(*b*) Infinity passes through a real dp and touches the quartic at an ordinary point. The dp may or may not lie on the circuit touched.

(*c*) Infinity joins two real distinct dps; i.e., distinct from each other, though one of them may be a tacnode or a node-cusp. The dps may be on the same circuit; on different circuits; one on one circuit, one common to this circuit and another: both common to two circuits.

[4] One 3-fold point and one 1-fold point.

(*a*) Infinity cuts the quartic once and is an inflexional tangent at another point. The inflexion and the ordinary intersection are on different circuits in 7, III and 8, 13, IV; otherwise they are on the same circuit.

(*b*) Infinity cuts the quartic and is tangent at a crunode or ordinary cusp; the ordinary intersections may or not be on the circuit touched.

(*c*) Infinity cuts the quartic and passes through a triple point.

[5] One 4-fold point.

(*a*) Infinity is an inflexional tangent at a flecnode (possible in all those types that may have a real inflexion capable of varying from one side of a crunode to the other.)

(*b*) Infinity is tangent at a tacnode, node-cusp, oscnode, or tacnode-cusp.

(*c*) Infinity is tangent to any real branch at a triple point.

For an exhaustive analysis these cases require further subdivision according to the nature of the dp or dps through which infinity passes, and according as real points of ordinary intersection with infinity are or are not in themselves points of inflexion. A classification that would approach the minutiæ of Newton's classification of cubics would give almost innumerable types of quartics.

Note A.

The proof given by Klein of his equation connecting the number of real singularities in the case where these are simple is purely geometric; the only consideration of the corresponding equation for composite singularities appears to be that given by Brill, already referred to, this depending on purely algebraic principles. There does not appear to have been any attempt made to determine by the principles of Analysis Situs how Klein's equation is to be interpreted in the case of composite singularities. For this reason it may be of interest to collect in this note the results already found regarding the interpretation of the equation for the compound singularities met with in quartic curves.—The table (Art. **2**) was constructed on the assumption that Klein's equation, *properly interpreted,* applies to compound singularities, though the argument nowhere relies on this assumption. It has been noted also that certain forms provided in the table do not exist, or exist only under certain conditions. These results are here collected.

Of the forms enumerated in the table, (5), II exists only in quartics in which the two crunodes are so specialized as to give an isolated tacnode or a pair of imaginary nodes (i.e., quartics in which the circuit is non-singular); or in quartics possessing a triple point of the fourth kind (in which case the circuit is non-singular to the eye).

The forms (15), (22), II, do not exist among quartics with *consecutive* dps.

The forms (9), (18), II; (7), (15), (19), IV do not exist at all.

From the present point of view the isolated tacnode is not the equivalent of two acnodes, although it is the limit of two acnodes moved up to consecutive position; it is, on the contrary, the equivalent of two crunodes, although crunodes, when real, are

usually thought of as formed by real branches. A similar thing is true for the isolated oscnode.

The triple point of the fourth kind is an especially interesting combination of dps: since it is formed by a real branch passing through an acnode, the latter is no longer an acnode under the definition *isolated point*, although it is still the intersection of conjugate imaginary branches; nor can the two nodes formed by the real branch crossing these imaginary branches be regarded as 'regulation' crunodes; we have found, however, that the triple point of the fourth kind does count as two crunodes and one acnode.

Thus, for the purposes of Klein's equation, the essential characteristics of a dp in order that it may count as an acnode in a quartic curve are that it be formed by the intersection of conjugate imaginary branches and that these have, at their intersection, conjugate imaginary tangents.

Note B.

Circuits are usually divided into even and odd, but as previously stated in Article **20** certain even circuits have long been recognized as endowed with a special property,—that of being met by every straight line in at least two real points. Thus the circuits of a quartic have here been named even, double-odd, and odd. The quartic may have even circuits in number not exceeding four; one double-odd circuit with or without an even circuit; or two odd circuits with or without an even circuit,—the number being in all cases controlled by Harnack's relation 'the maximum number of circuits is one more than the deficiency.'

Note C.

In the *Zeitschrift für Mathematik und Physik*, XXXV, pp. 25–35 (1890), Binder discusses the dts of certain divisions of unicursal quartics, with reference to the reality of the dts and of their points of contact. He calls attention to an error made by Ameseder in stating that, if all four dts of a quartic with three nodes be real, one must be isolated. It is perhaps well to guard against too broad an interpretation of some of Binder's own formal statements. It is stated on page 26 (*loc. cit.*) that the four dts of a

quartic with three dps, of which two may form a tacnode, can all be real with real contacts; *Doppelpunkt* is evidently to be translated *crunode;* when all three dps are acnodes one of the four dts is necessarily isolated; further, when the quartic is a double-odd circuit the only real dts possible are the two at a tacnode, if there be one. On page 34 it is stated that a quartic with a tacnode and another dp, and with two 'proper' dts (distinct dts with real contacts) has exactly two real inflexions: here, also, *Doppelpunkt* is to be translated *crunode;* when the distinct dp is an acnode and the quartic has two distinct dts with real contacts, the curve presents *four* real inflexions.

It is perhaps as well to call attention also to a slight inaccuracy on page 25, in the statement of known results regarding the dts of a unicursal quartic; the wording implies that the division of the dts into real and imaginary depends on the division of the dps into crunodes and acnodes (the word *cusps*, occurring in the paragraph, is evidently a slip); as a matter of fact the difference is caused not by the nature of the *dps*, but by the nature of the *circuit;* the dependence for unicursal quartics may be formulated as follows: When the circuit is even (see Note B) the dts are all real except in the case of the quartic with a triple point of the fourth kind, in which case two dts are real, two imaginary; when the quartic has a double-odd circuit it can have no real dt except when there is a tacnode, in which case the only real dts are those at the tacnode.

Again, in the quartics referred to in the next sentence (in which two of the three nodes form a tacnode), the number of dts distinct from those at the tacnode is of course *precisely* two, not simply *höchstenfalls* two; the writer evidently refers to the number of *non-isolated* dts apart from those at the tacnode.

Plates.

The figures of Plates I–IV, P, with the exception of perhaps a dozen, copied from Zeuthen (*loc. cit.*) and Salmon (Higher Plane Curves), are only rough drawings intended to illustrate the salient features of the types. For instance, often in Plates I, III, IV some or all of the ovals would probably be impossible with the remainder of the figure as drawn, but they are inserted to mark the divisions of the plane in which they belong when the constants are such that the ovals do all exist. This is always possible for some quartic of the type, but would sometimes necessitate a very small scale in order to bring the entire figure into the finite part of the plane within reasonable limits, would sometimes make the bays almost invisible, and would sometimes yield both these undesirable results. The inflexional tangents at the two inflexions on the same bay must lie on the same side of another real circuit of the quartic as does the dt bridging that bay ; where this is not so shown it is inaccuracy of the figure either accidental or serving to make the bay readily apparent. Similarly, the inflexional tangent at a single inflexion lying between a dp and the contact of a tangent *from* that dp to the same circuit must lie on the same side of another real circuit as does the abnodal tangent; *etc.* But although the figures are only approximate, still a certain measure of accuracy has been aimed at : the bays are faced in the right direction with reference to other circuits, the sharpness of contour of an embayed circuit in the neighborhood of an oval is intentional, *etc.;* for a figure in which a particular oval is not present an embayed circuit may be much more rounded out toward the triangle or quadrilateral in which an oval belongs when the maximum number of circuits are present, and may thus be also more deeply embayed. Again, a certain approach to symmetry is often observable in the figures ; this is of course not an essential feature.

The small bracketed numeral following the number of a figure indicates the place of the figure in the table (Art. **2**).

The letter *I*, appearing in certain figures of Plate II, emphasizes the presence of a real inflexion necessary to the existence of the type. It has not been thought necessary to insert this letter in the derived figures of Plates III, IV, and P.

PLATE I.

FIG. 1 (1)

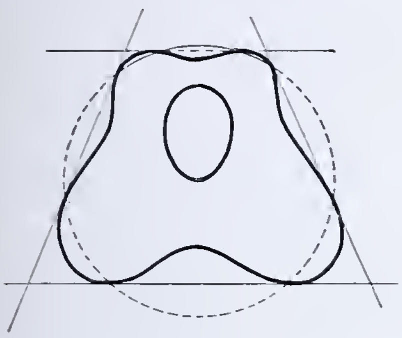

FIG. 2, 3. (1)

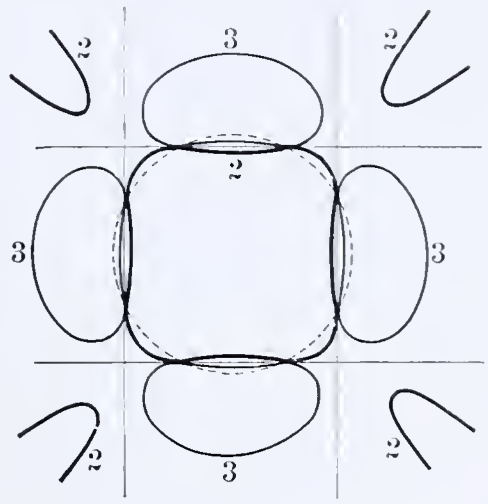

FIG. 2'. (1)

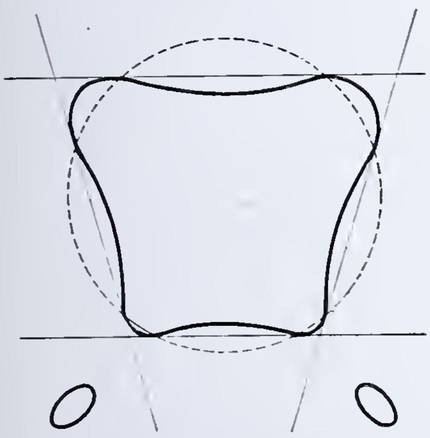

FIG. 4, 5. (1)

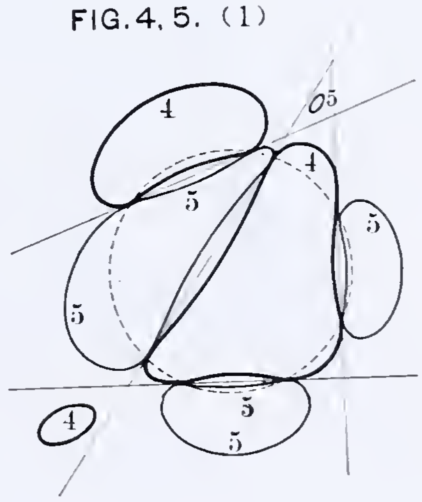

FIG. 6'. (1)

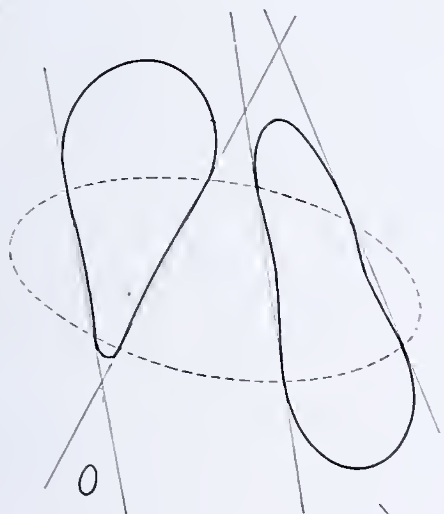

FIG. 6, 7. (1)

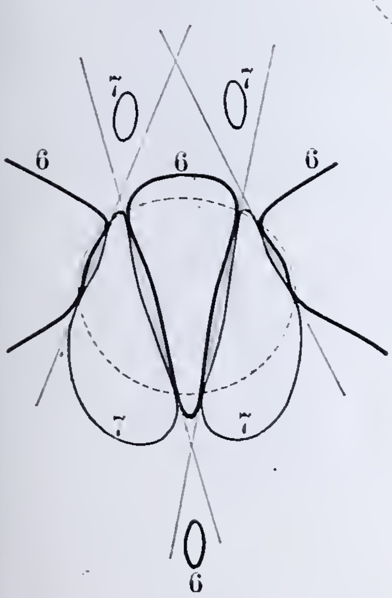

FIG. 8, 9. (1)

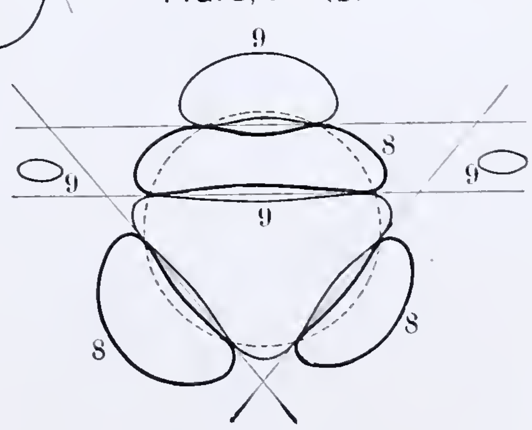

PLATE I

FIG. 10. (2)

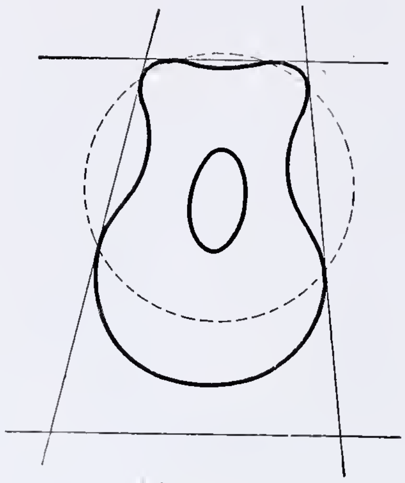

FIG. 11, 12. (2)

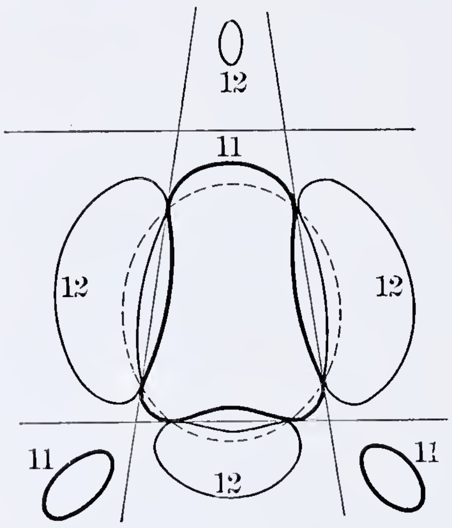

FIG. 13, 14. (2)

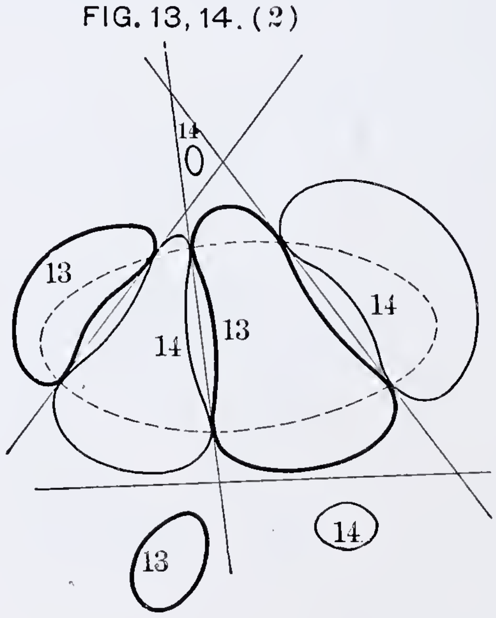

FIG. 15, 16. (2)

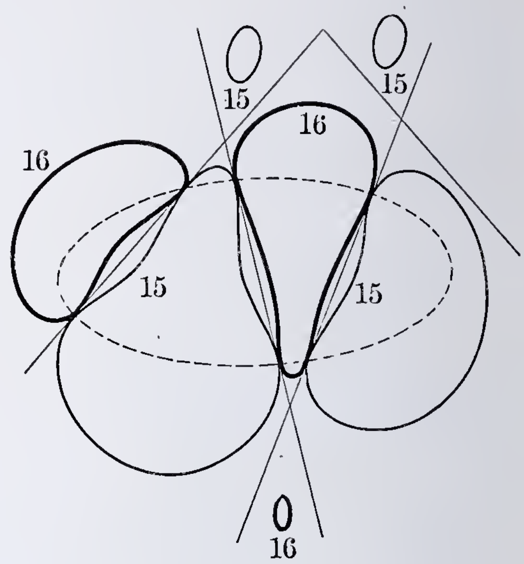

FIG. 17. (2)

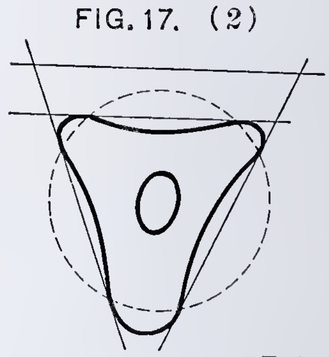

FIG. 18, 19. (2)

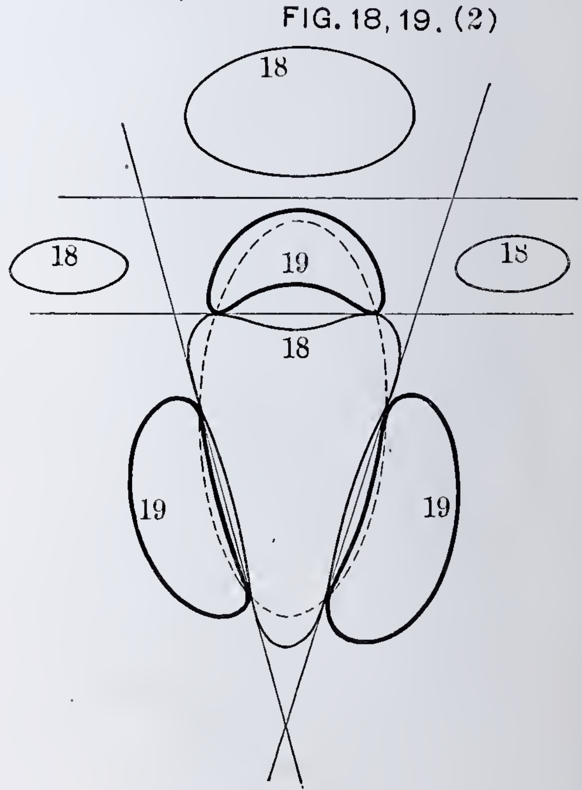

PLATE I

FIG. 20. (3)

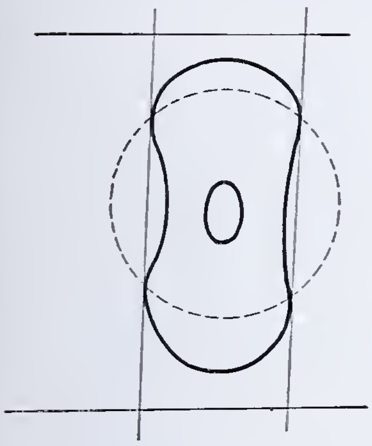

FIG. 21, 22. (3)

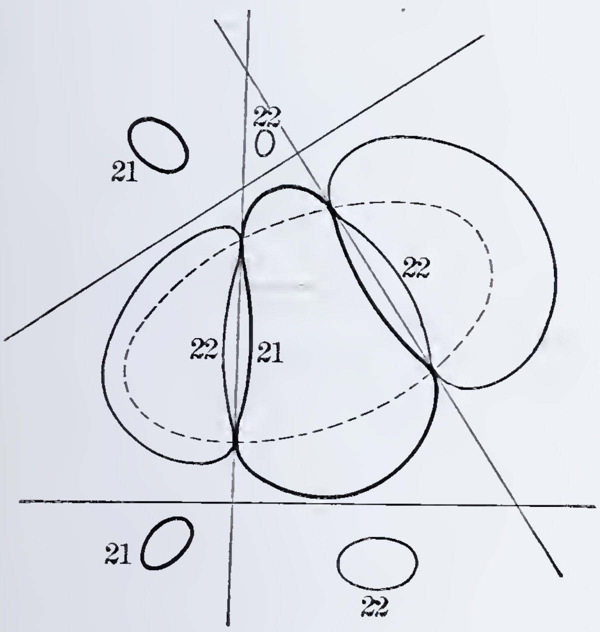

FIG. 23. (3)

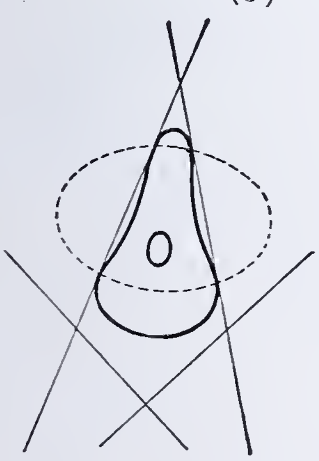

FIG. 24, 25. (3)

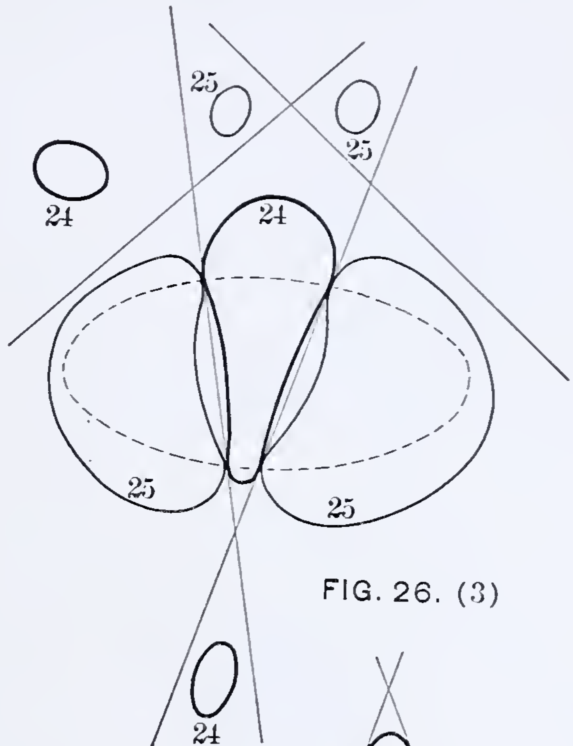

FIG. 26. (3)

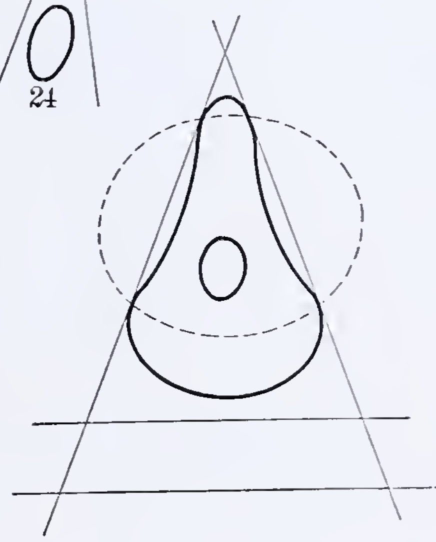

FIG. 27, 28. (3)

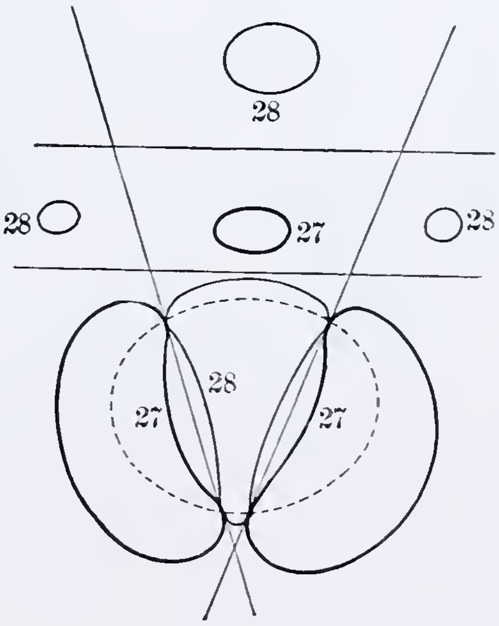

PLATE I

FIG. 29. (4)

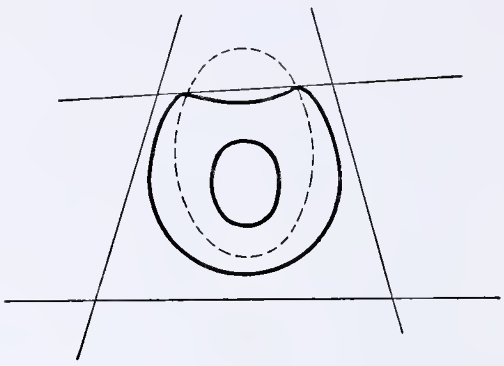

FIG. 30. (4)

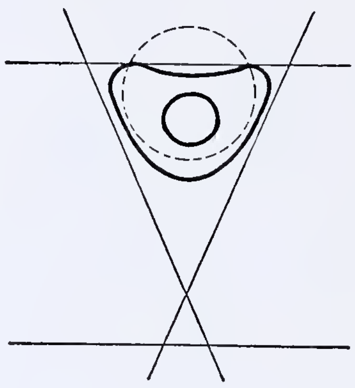

FIG. 31, 32. (4)

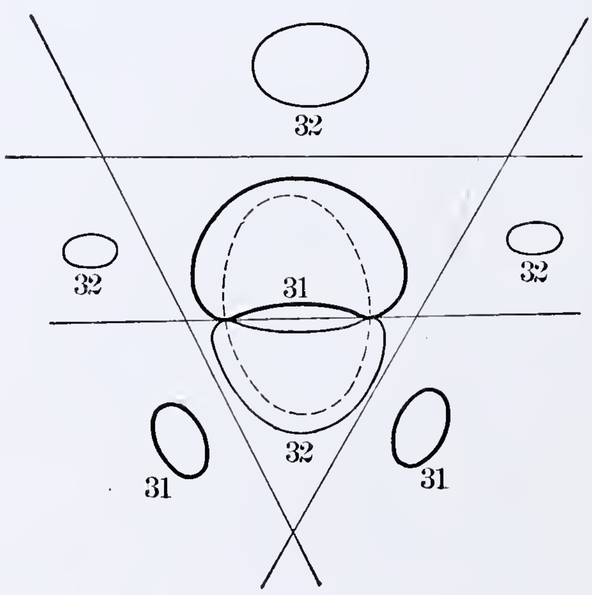

FIG. 33. (5)

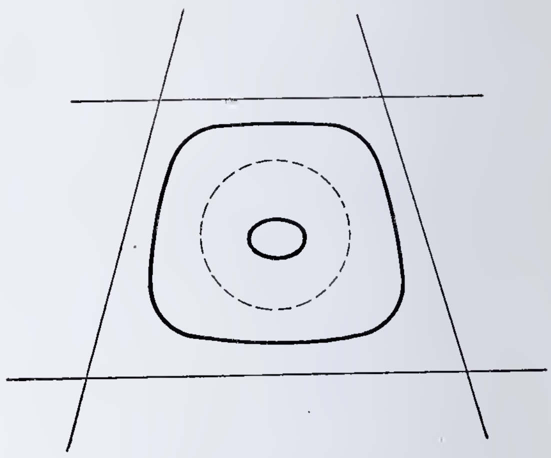

FIG. 34. (5)

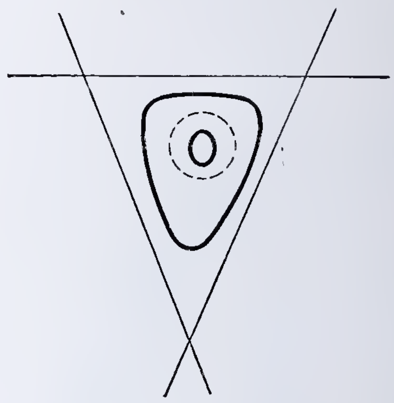

FIG. 35, 36. (5)

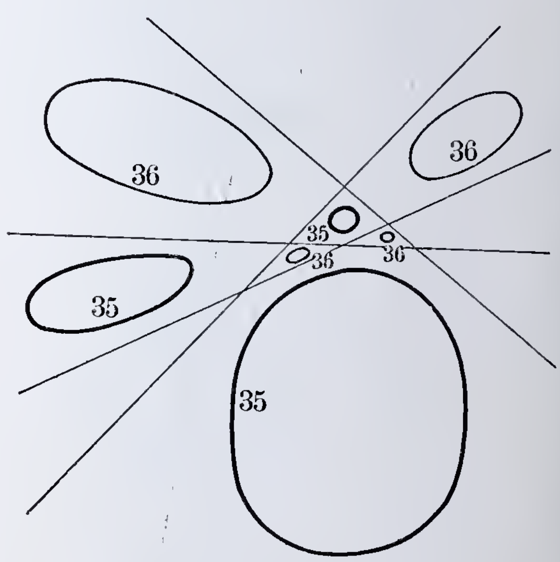

PLATE II

FIG. 1_c. (2)

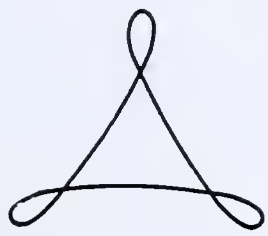

FIG. 1_a. (1)

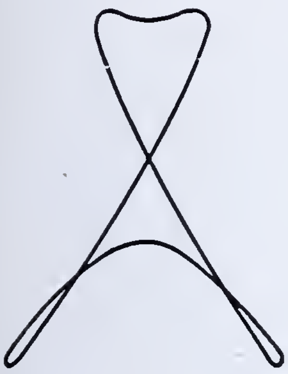

FIG. 1_b. (1)

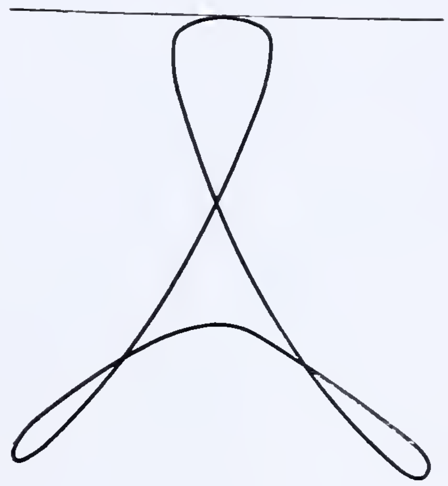

FIG. 2. (1)

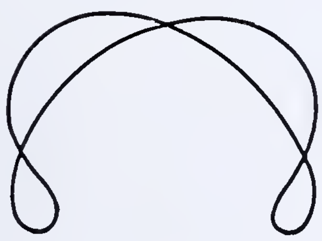

FIG. 3_a. (1)

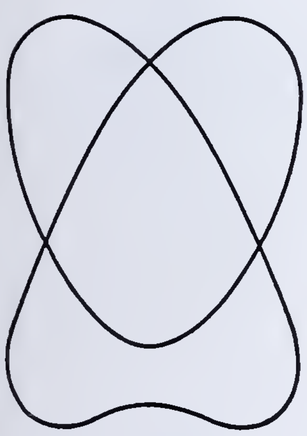

FIG. 3_b. (1)

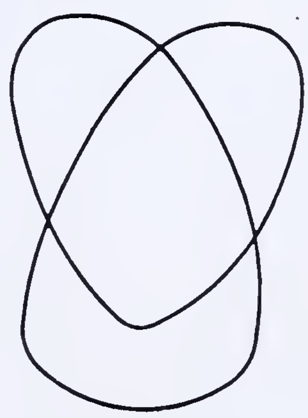

FIG. 3_c. (2)

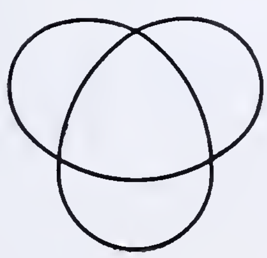

PLATE II

FIG. 4_a. (1)

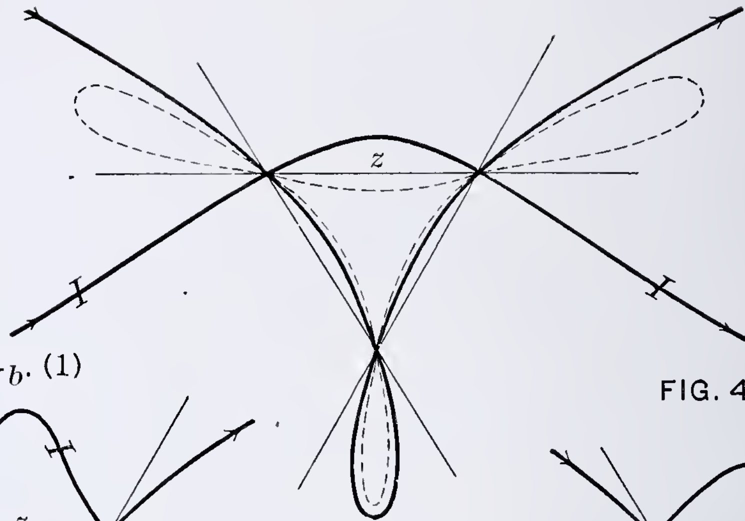

FIG. 4_b. (1)

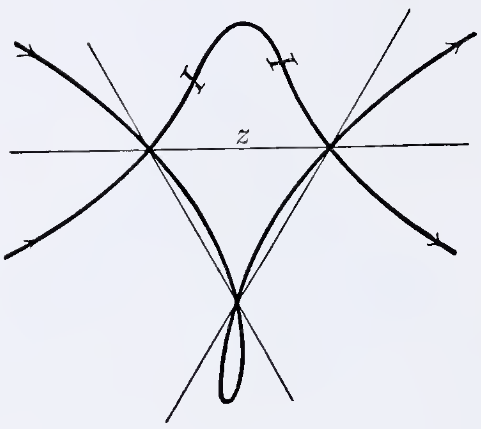

FIG. 4_c. (1)

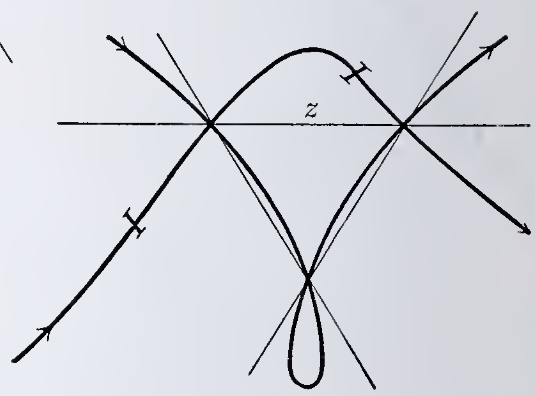

FIG. 5_b. (1)

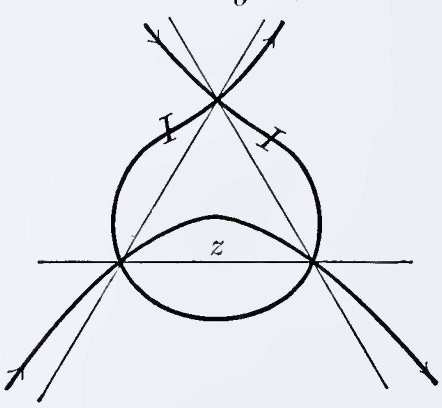

FIG. 5_c. (1)

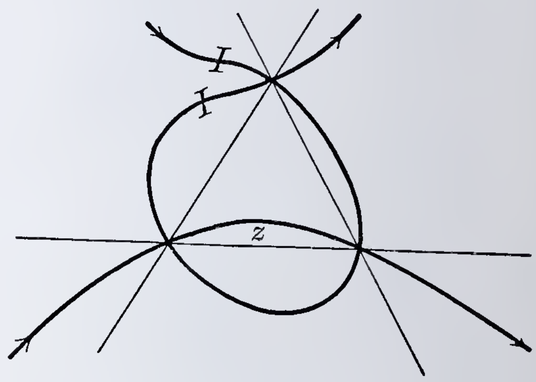

FIG. 5_a. (1)

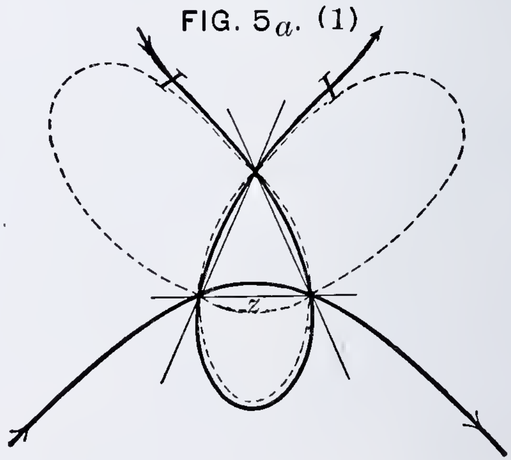

PLATE II

FIG. 6a. (14)

FIG. 6b. (15)

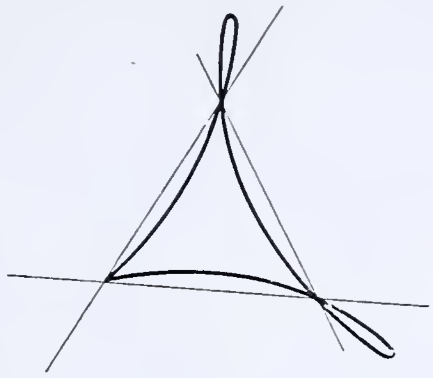

FIG. 7. (14)

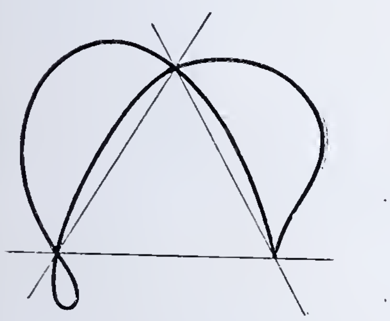

FIG. 8. (14)

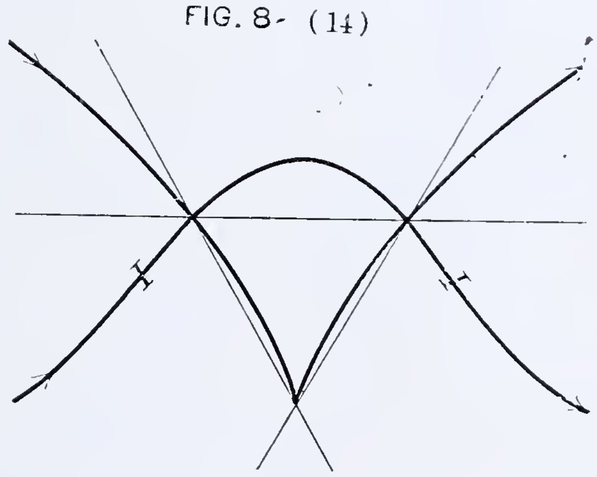

FIG. 9a. (21)

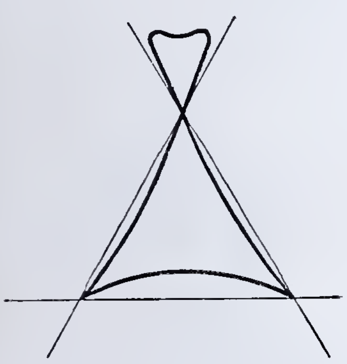

FIG. 9b. (22)

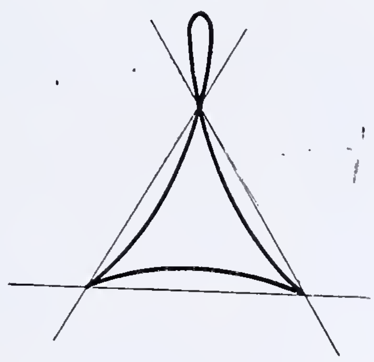

FIG. 10. (21)

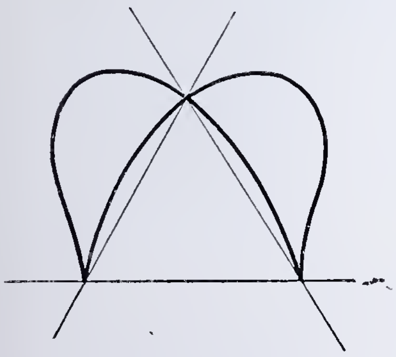

FIG. 11. (27)

PLATE II

FIG. 12$_{b}$. (4)

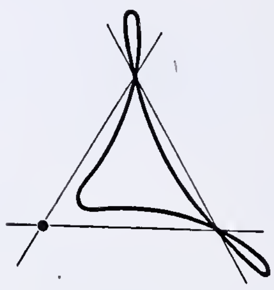

FIG. 12$_{c}$. (3)

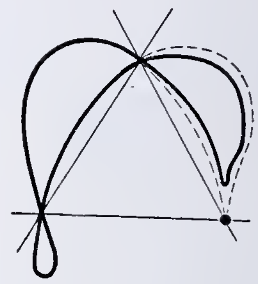

FIG. 12$_{a}$. (3)

FIG. 12$_{d}$. (4)

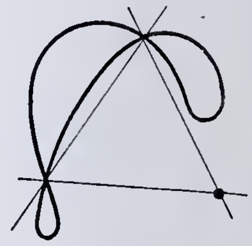

FIG. 13. (3)

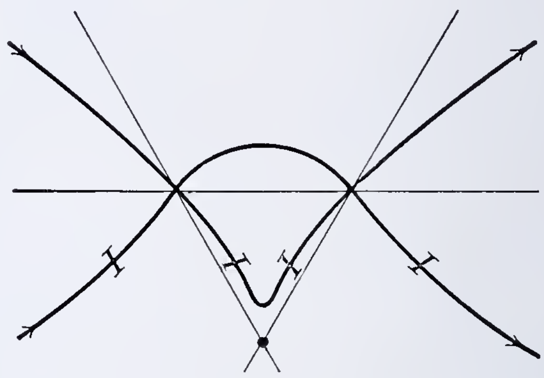

FIG. 14$_{a}$. (16)

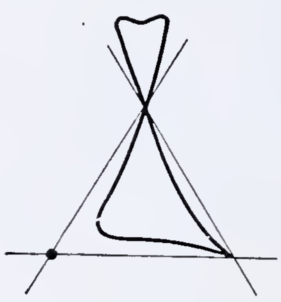

FIG. 15. (23)

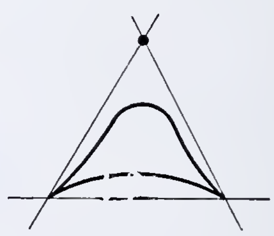

FIG. 14$_{d}$. (17)

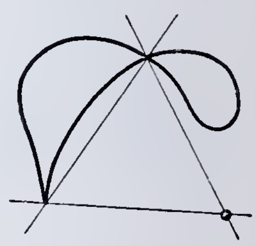

FIG. 14$_{b}$. (17)

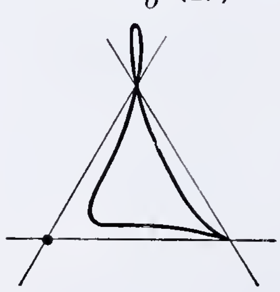

FIG. 14$_{c}$. (16)

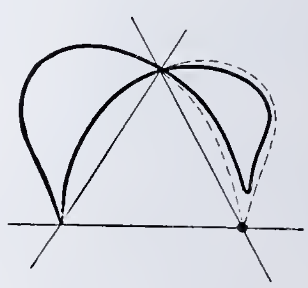

PLATE II.

FIG. 16*a*. (6)

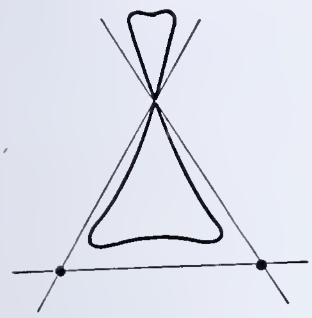

FIG. 16*b*. (7)

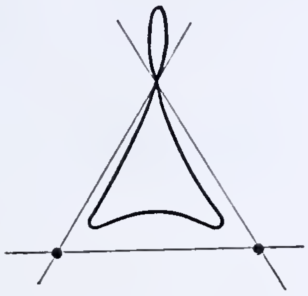

FIG. 16*c*. (7)

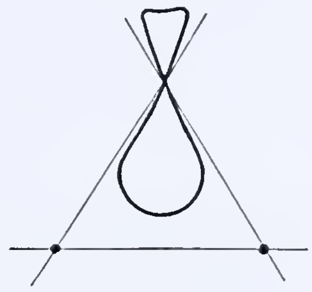

FIG. 16*d*. (8)

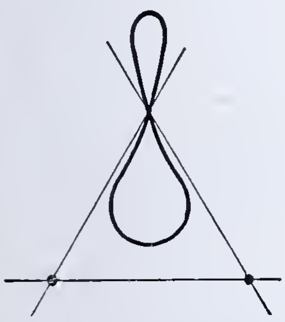

FIG. 16*e*. (6)

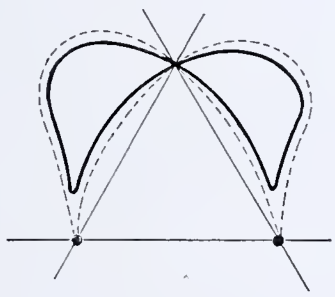

FIG. 16*f*. (7)

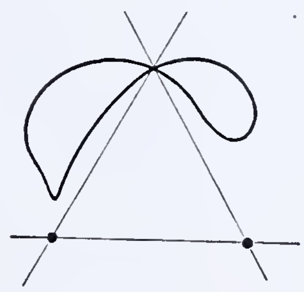

FIG. 16*g*. (8)

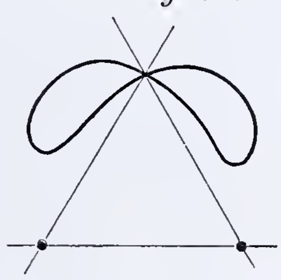

FIG. 17*a*. (19)

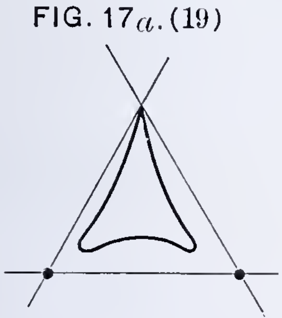

FIG. 17*b*. (20)

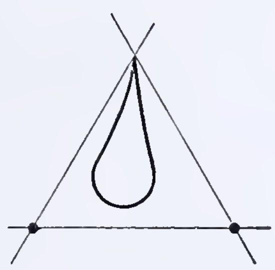

FIG. 18*a*. (10)

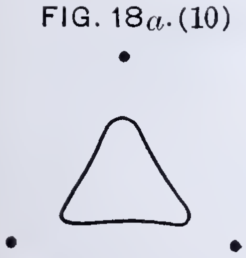

FIG. 18*b*. (11)

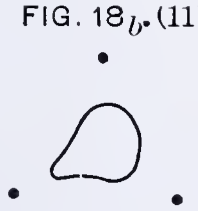

FIG. 18*c*. (12)

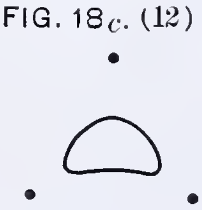

FIG. 18*d*. (13)

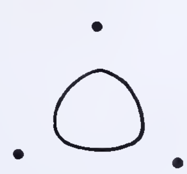

PLATE II

FIG. 19. (1)

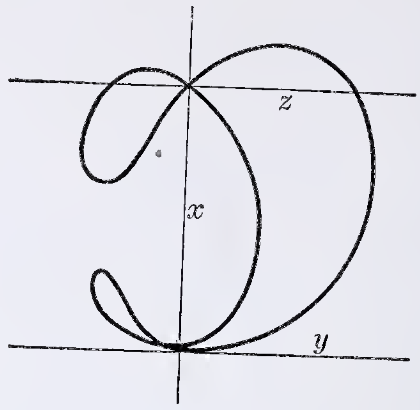

FIG. 20_a. (1)

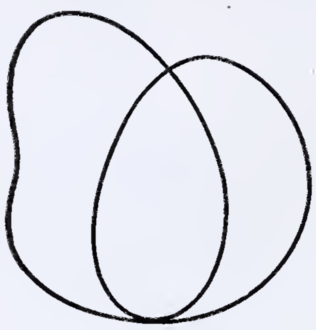

FIG. 20_b. (2)

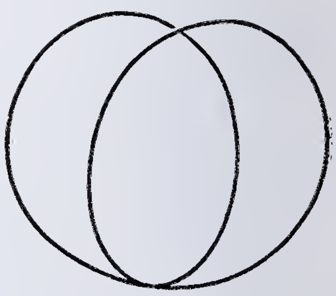

FIG. 21. (14)

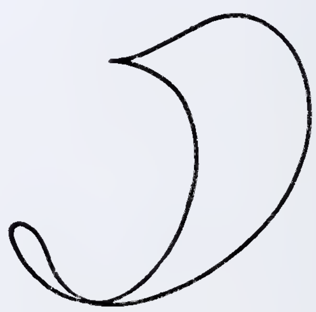

FIG. 22_a. (3)

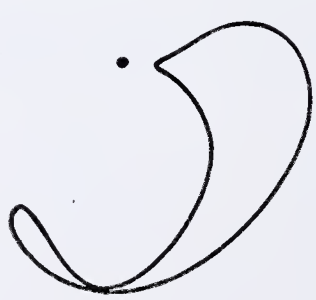

FIG. 22_b. (4)

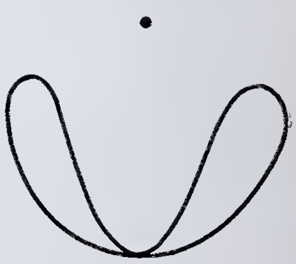

PLATE II

FIG. 23. (1)

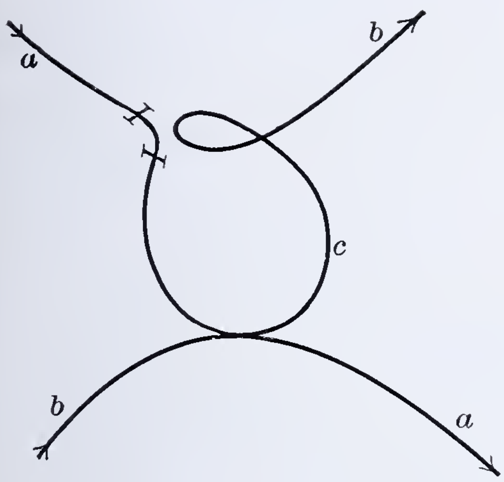

FIG. 24_c. (1)

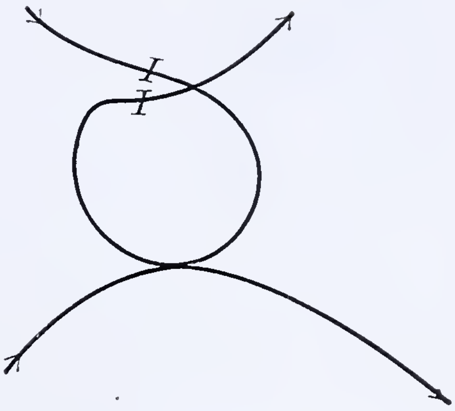

FIG. 24_a. (1)

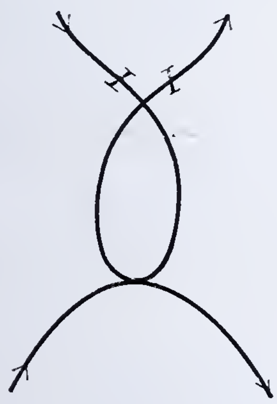

FIG. 25. (14)

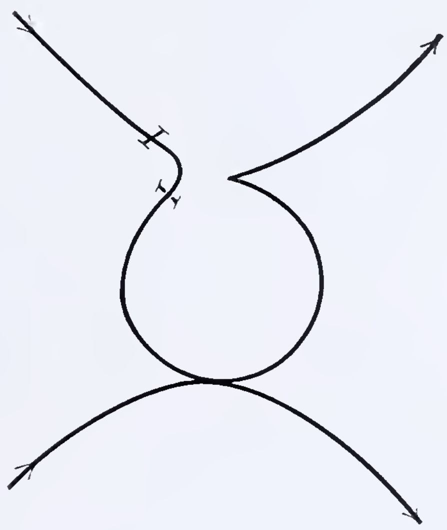

FIG. 24_b. (1)

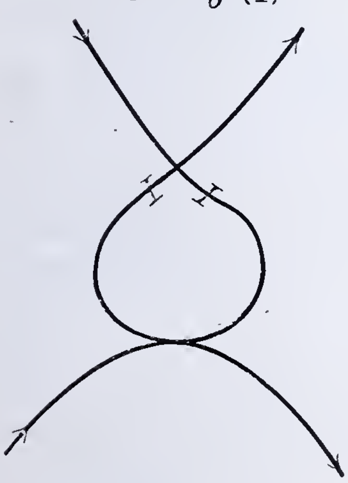

FIG. 26. (3)

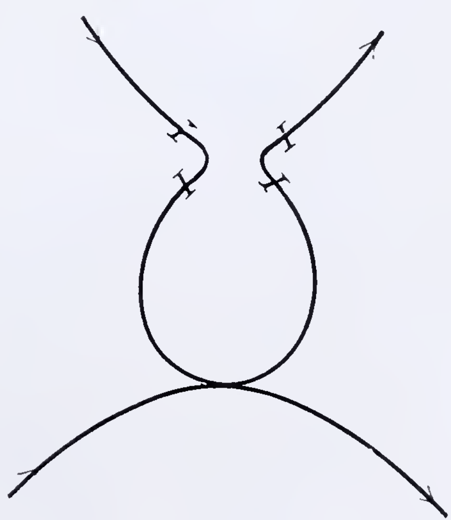

PLATE II

FIG. 27_b. (1)

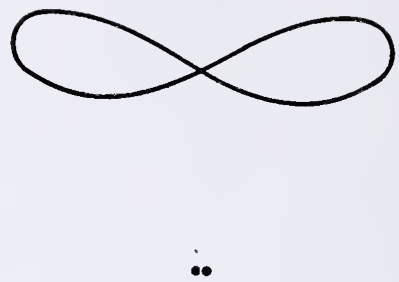

FIG. 27_a. (1)

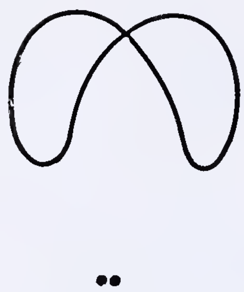

FIG. 27_c. (1)

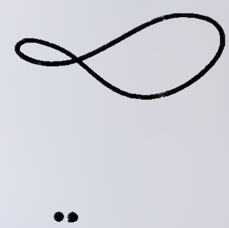

FIG. 28. (14)

FIG. 29_a. (3)

FIG. 29_c. (5)

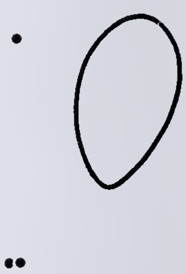

FIG. 29_b. (4)

FIG. 30. (14)

FIG. 31. (21)

FIG. 32_a. (16)

FIG. 32_b. (17)

PLATE II

FIG. 33_a·(1)

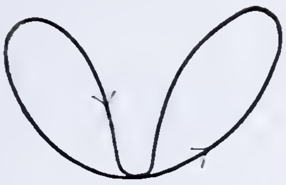

FIG. 33_b·(1)

FIG. 34_a·(1)

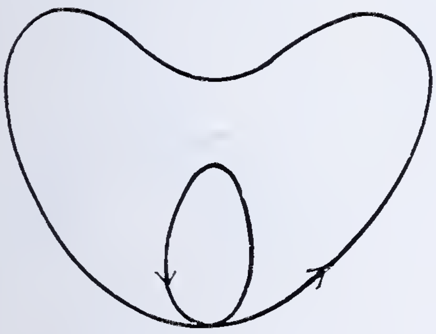

FIG. 34_b·(1)

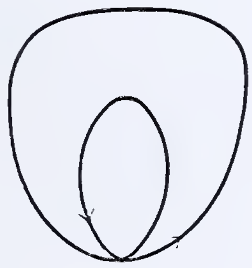

FIG. 34_c·(2)

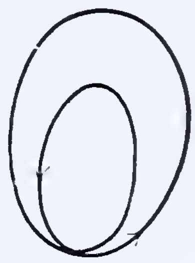

FIG. 35_a·(1)

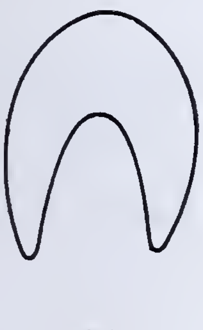

FIG. 35_b·(1)

FIG. 35_c·(2)

FIG. 36_a· (14)

FIG. 36_b· (14)

PLATE II

FIG. 37a. (1)

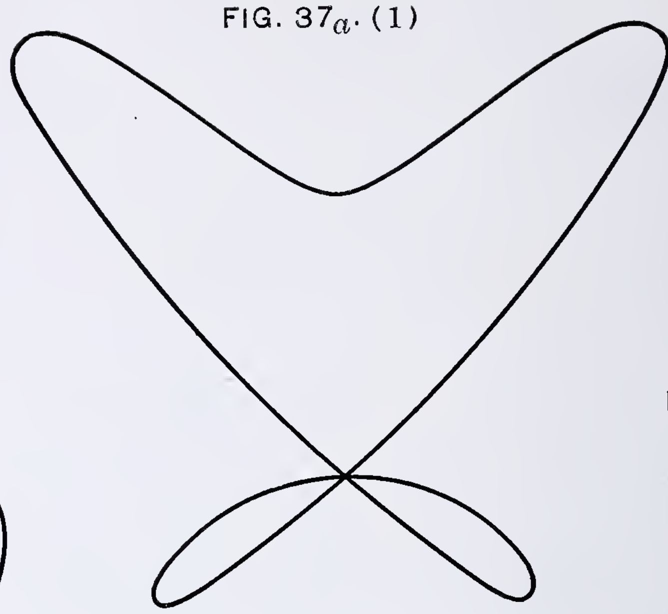

FIG. 37b. (1)

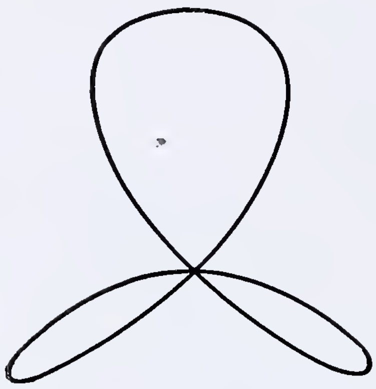

FIG. 37c. (2)

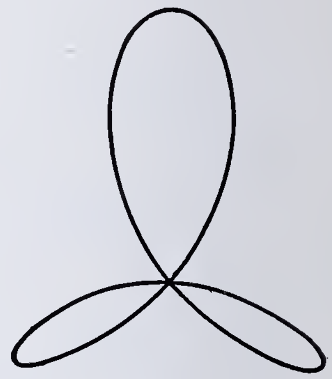

FIG. 38. (14)

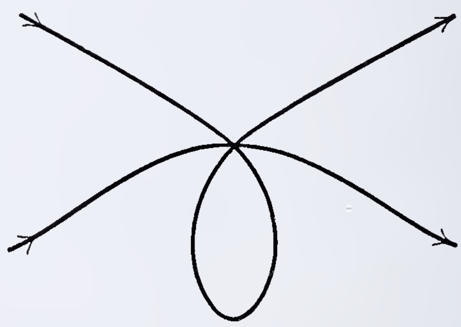

FIG. 39a. (14)

FIG. 39b. (15)

FIG. 40. (14)

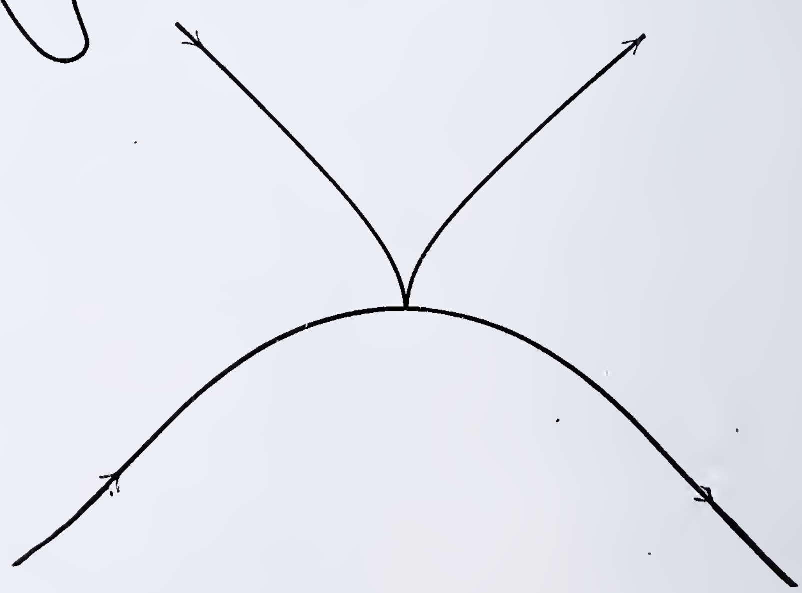

PLATE II

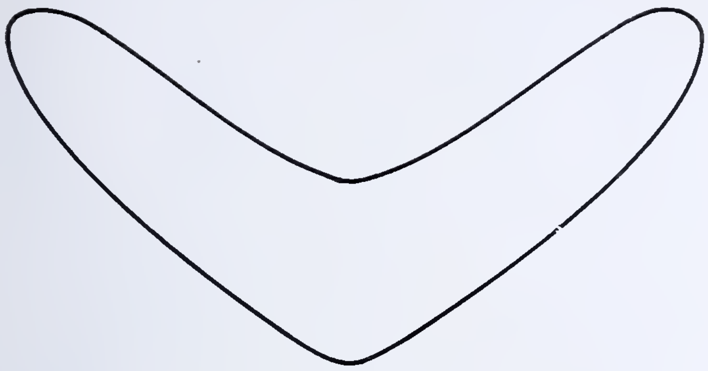

FIG. 41*a*. (21)

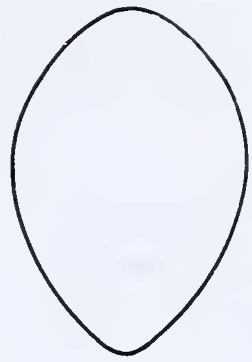

FIG. 41*b*. (22)

FIG. 42*a*. (3)

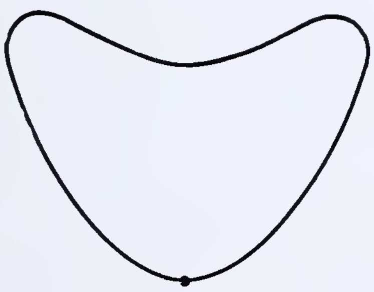

FIG. 42*b*. (4)

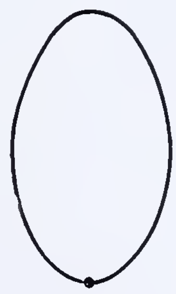

FIG. 42*c*. (5)

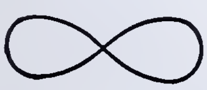

FIG. 43. (1)

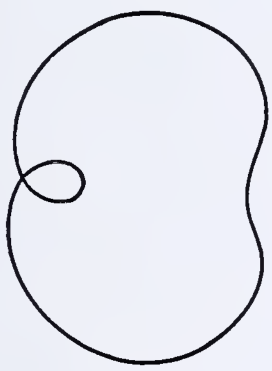

FIG. 44*a*. (1)

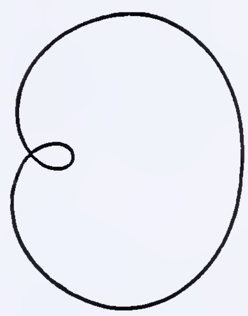

FIG. 44*b*. (2)

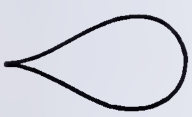

FIG. 45. (14)

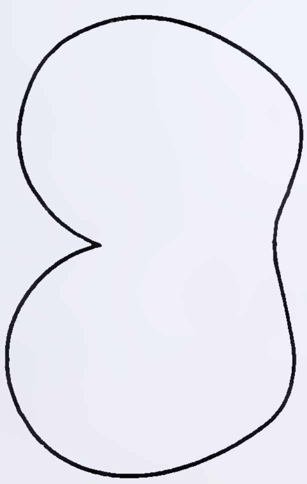

FIG 46*a*. (14)

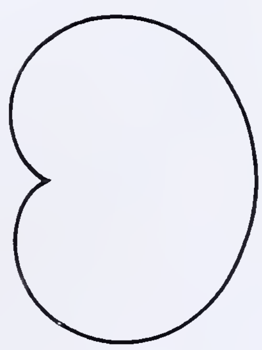

FIG. 46*b*. (15

PLATE II.

FIG. 47*a*. (3)

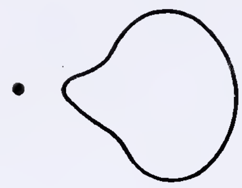

FIG. 47*b*. (4)

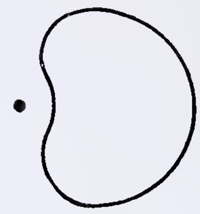

FIG. 47*c*. (5)

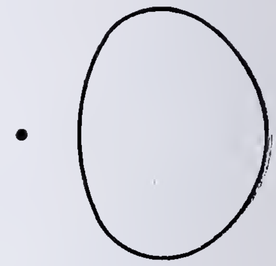

FIG. 48*a*. (3)

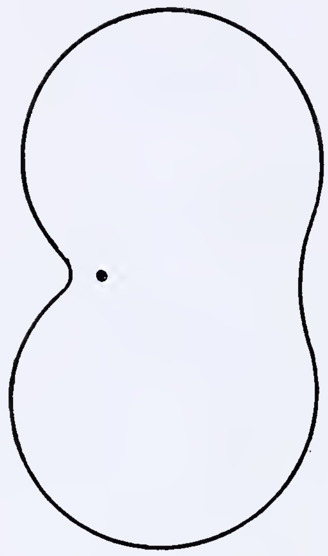

FIG. 48*b*. (4)

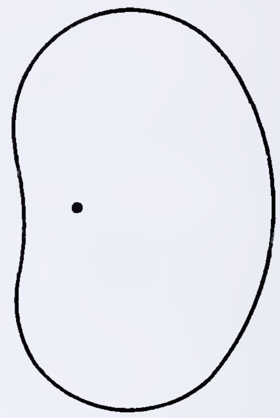

FIG. 48*c*. (5)

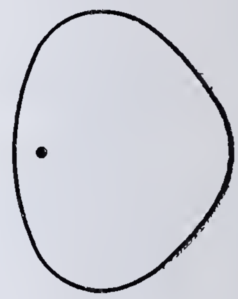

FIG. 50. (28)

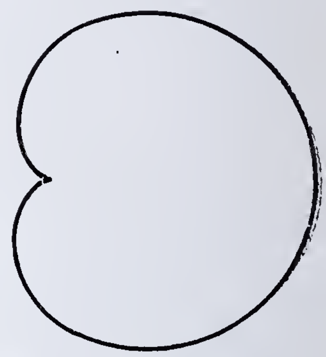

FIG. 49. (24)

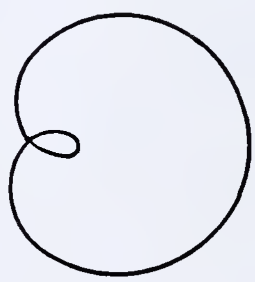

FIG. 51*b*. (26)

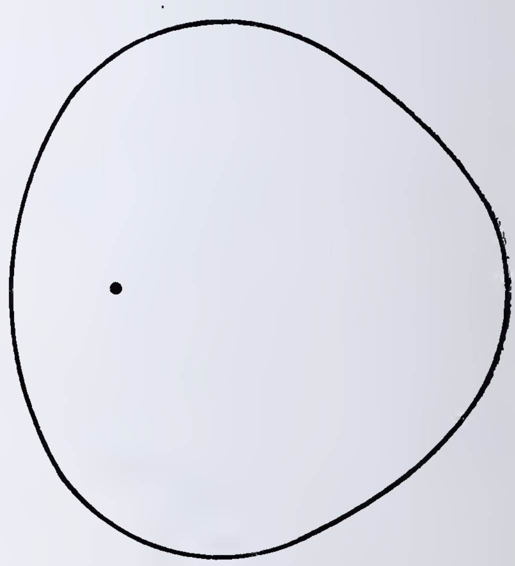

FIG. 51*a*. (25)

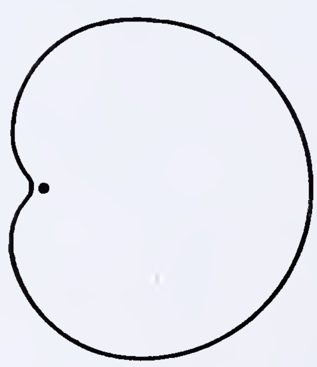

PLATE III

FIG. 1. (1)

FIG. 2. (1)

FIG. 3. (1)

FIG. 4. (1)

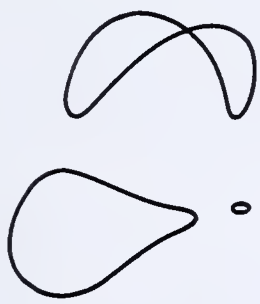

FIG. 5. (1)

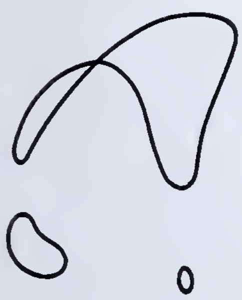

FIG. 6. (1)

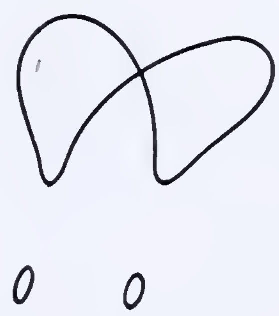

FIG. 7. (1)

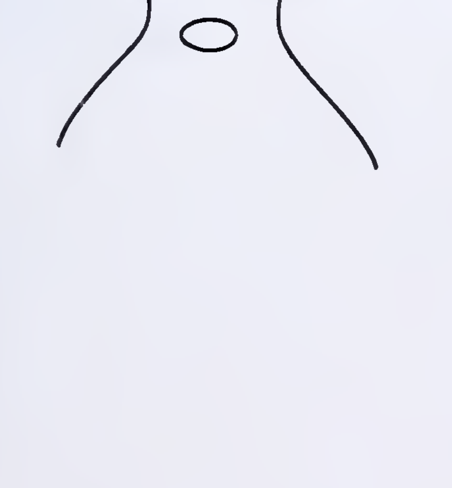

PLATE III

FIG. 8. (10)

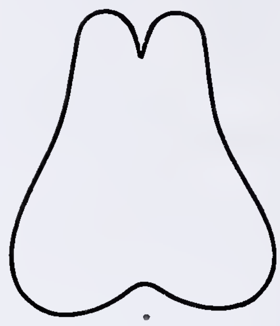

FIG. 10. (10)

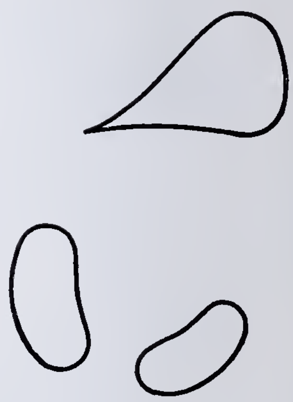

FIG. 9. (10)

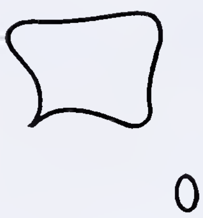

FIG. 11. (10)

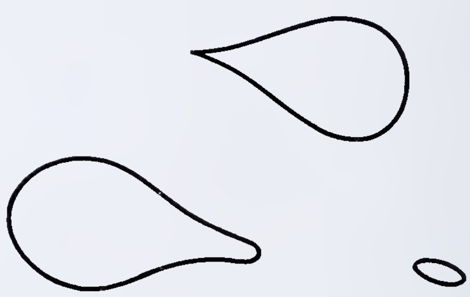

FIG. 12. (10)

FIGS. 13-17
FOR QUARTICS WITH ONE ACNODE,
13-17, SEE ART. 77

PLATE IV

FIG. 1. (1)

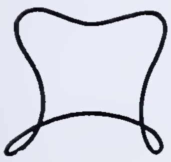

FIG. 2. (1)

FIG. 3. (1)

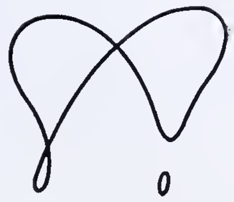

FIG. 4. (1)

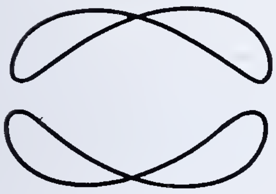

FIG. 5. (1)

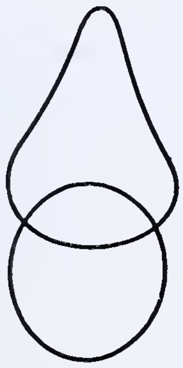

FIG. 6. (1)

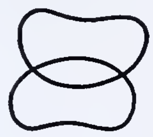

FIG. 7. (1)

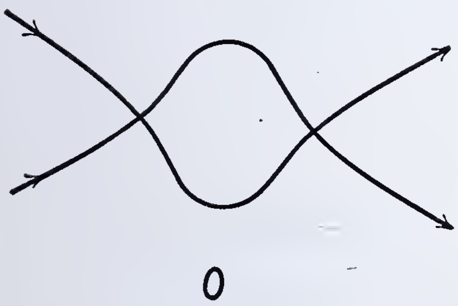

FIG. 8. (1)

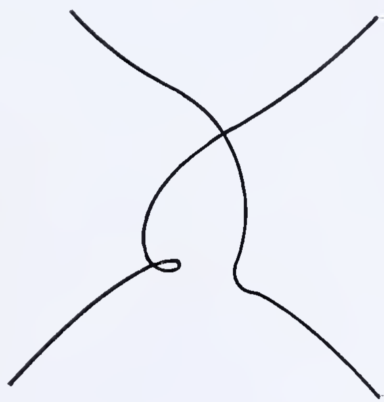

FIG. 9. (13)

FIG. 10. (13)

FIG. 11. (13)

FIG. 12. (13)

FIG. 13. (13)

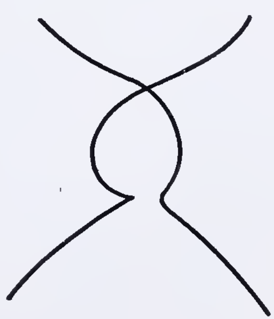

PLATE IV

FIG. 14. (20)

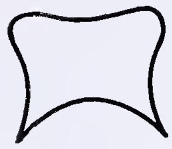

FIG. 15. (20)

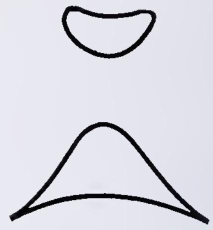

FIG. 16. (20)

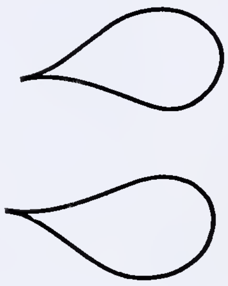

FIG. 17–27,

FOR QUARTICS WITH AN ACNODE AND
A CRUNODE, 17–21,
A CUSP, 22–24,
A SECOND ACNODE, 25–27,

SEE ARTS. 81, 82, 83.

FIG. 28. (1)

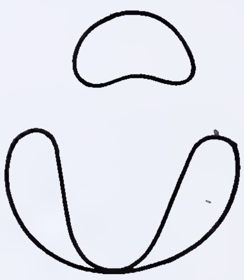

FIG. 29. (1)

FIG. 30. (1)

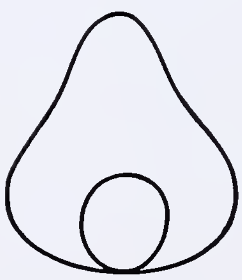

FIG. 31. (1)

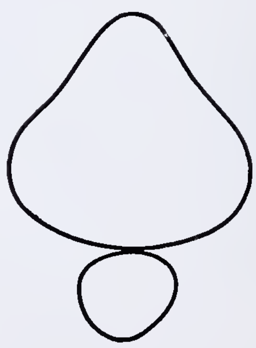

FIG. 32 (1)

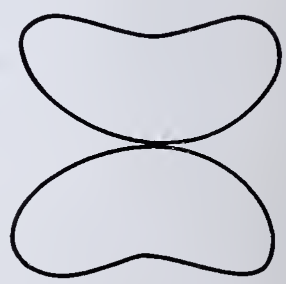

PLATE IV.

FIG. 33. (1)

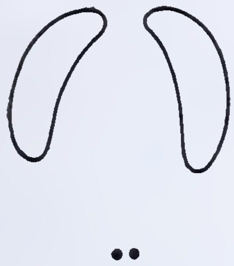

FIG. 34. (1)

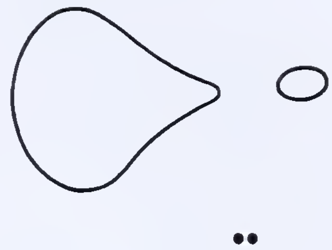

FIG. 35. (13)

FIG. 36. (13)

FIG. 37. (1)

FIG. 38. (1)

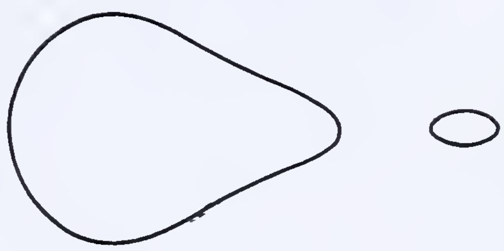

FIG. 39. (1)

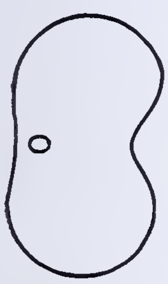

FIG. 40. (22)

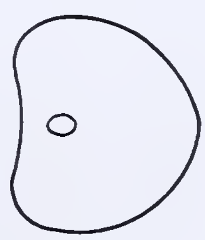

PLATE P

FIG. 1'.(1)

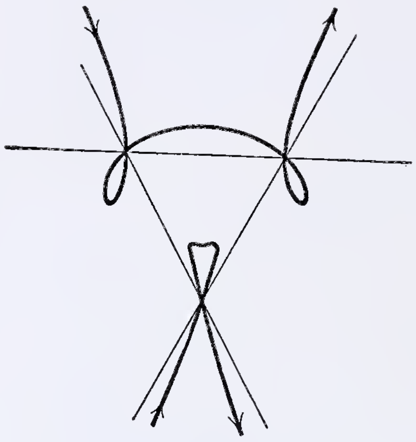

FIG. 1''.(1)

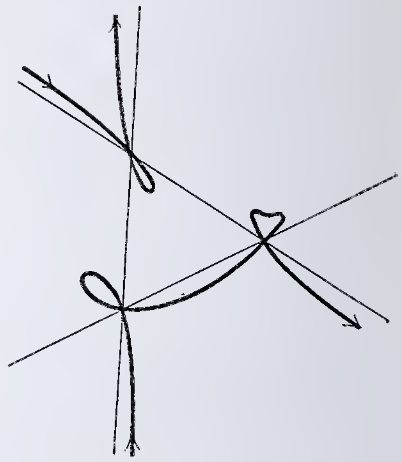

FIG. 2'.(1)

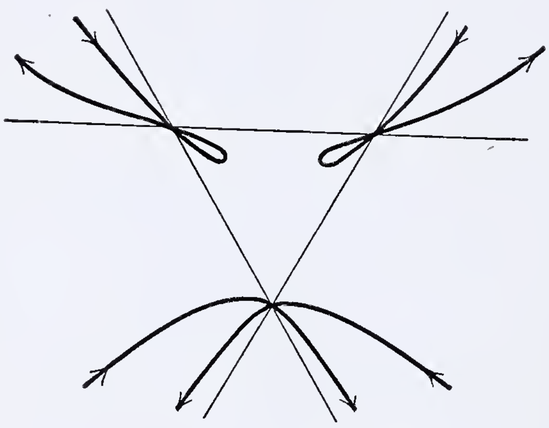

FIG. 2''.(1)

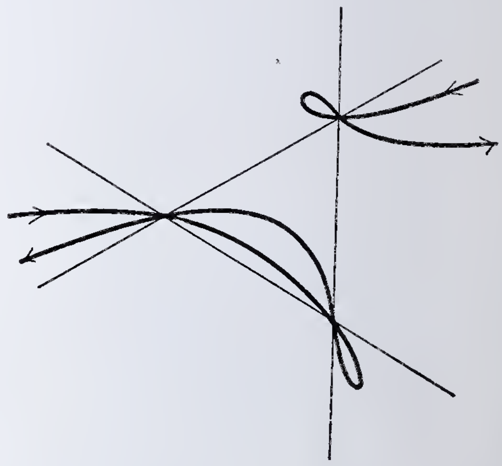

FIG. 3'.(2)

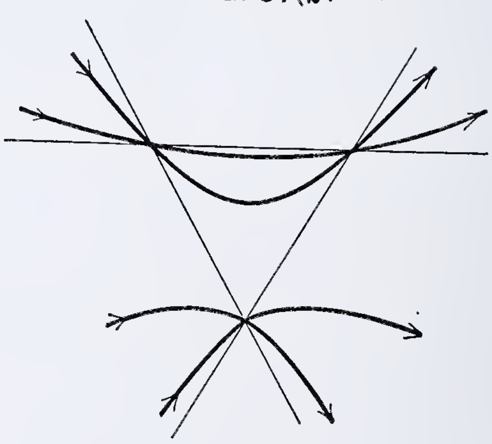

PLATE P

FIG. $4'_a$ (1)

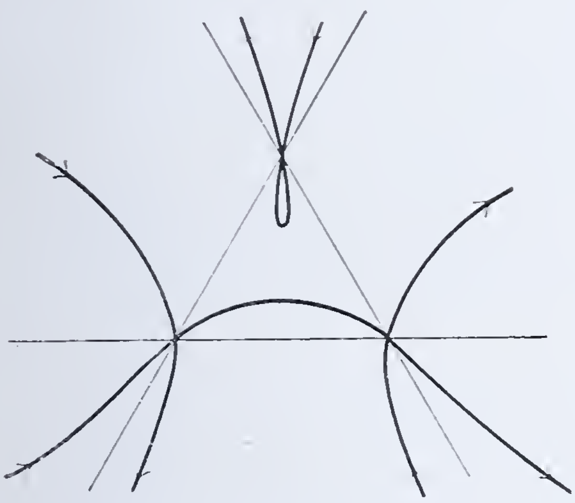

FIG. $4''_a$. (1)

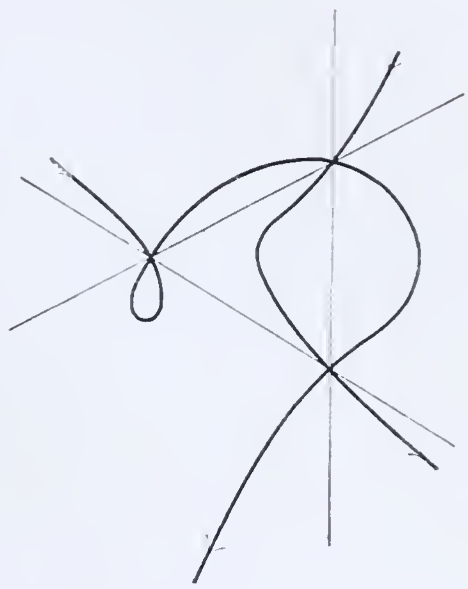

FIG. $4'_b$. (1)

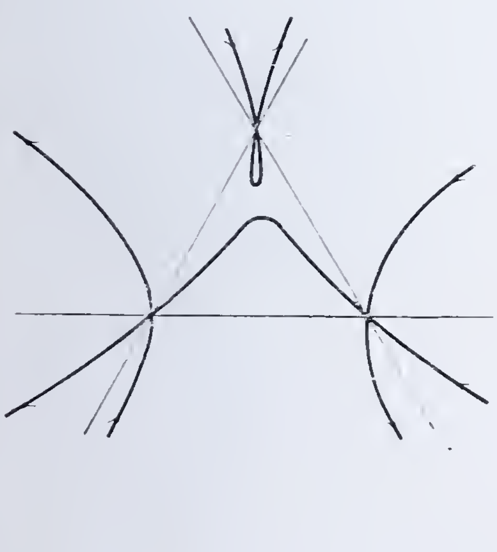

FIG. $4''_b$. (1)

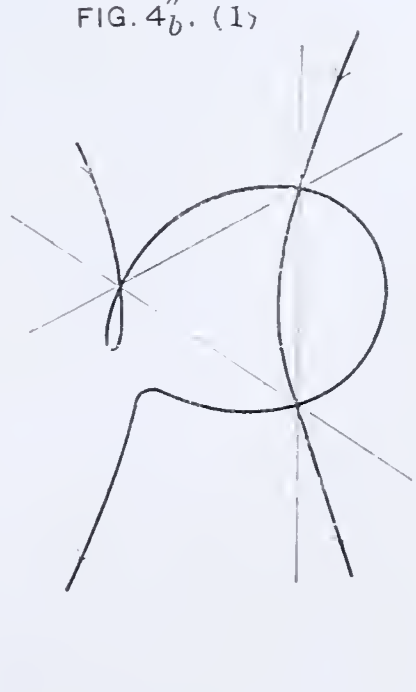

PLATE P

FIG. $5'_a$ *or* $5'_b$.(1)

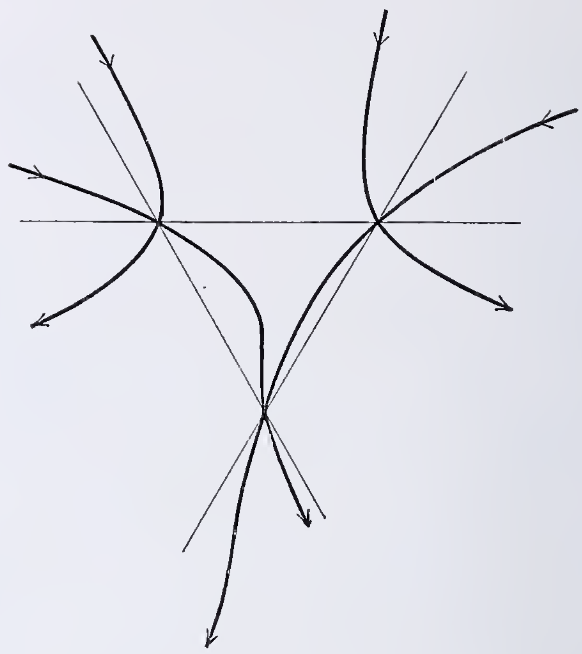

FIG. $5'_c$.(1)

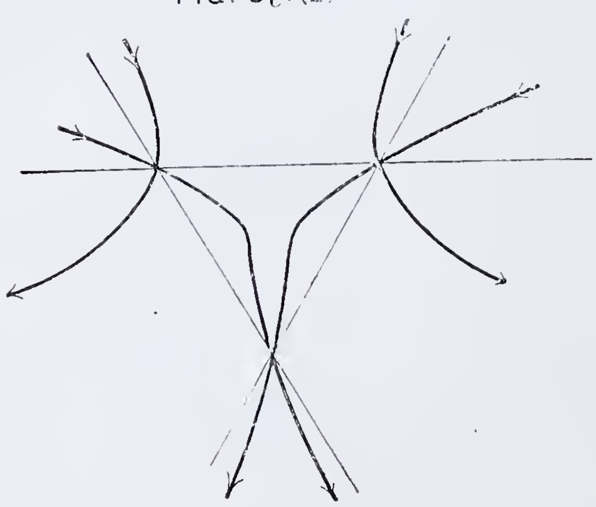

FIG. $5''_c$.(1)